

Geometric and non-geometric compactifications of IIB supergravity

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ABSTRACT: Complementary geometric and non-geometric consistent reductions of IIB supergravity are studied. The geometric reductions on the identified group manifold $\mathcal{X} = G/\Gamma$ are found to have a gauge symmetry with Lie algebroid structure, generalising that found in similar reductions of the Bosonic string theory and eleven-dimensional supergravity. Examples of such compactifications are considered and the symmetry breaking in each case is analysed. Complementary to the reductions on \mathcal{X} are the nine-dimensional S-duality twisted reductions considered in the second half of the paper. The general reduced theory is given and symmetry breaking is investigated. The non-geometric S-duality twisted reductions and their relation to geometric reductions of F-Theory on \mathcal{X} is briefly discussed.

KEYWORDS: Flux compactifications, Superstring Vacua, String Duality.

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1. Introduction

The majority of techniques used to study the dimensional reduction of $D + d$ dimensional Supergravity theories to D space-time dimensions may be divided into two broad categories.

The first, and most amenable to physical interpretation, requires the higher-dimensional theory to be rewritten in terms of the harmonic modes of some compact internal manifold and gives rise to an infinite tower of massive Kaluza-Klein fields corresponding to excitations of the higher harmonics. This results in a rather cumbersome

rewriting of the original theory and in practice only a truncation of the theory to the lightest modes is considered [1]. The drawbacks of this approach are that the treatment of the harmonic modes can be laborious in practice and the low energy modes that remain in the truncated theory generally will not solve the higher-dimensional equations of motion. Thus solutions of the truncated theory generally will not lift to solutions of the higher dimensional theory. This technique will be referred to as Kaluza-Klein reduction. The mass of the modes in the Kaluza-Klein reduction are given by the eigenvalues of the Laplacian on the internal manifold. As such the truncation to the lowest modes keeps the lightest states whose number is given by the Betti numbers of the internal manifold.

The second approach, and the one that will be considered in this paper, is to consider solutions of the higher dimensional equations of motion that may be given a lower dimensional *interpretation*. This lower dimensional interpretation arises from the requirement that the solution does not depend on the internal coordinates and generally will not involve all of the fields in the higher dimensional theory. Such a truncation of the spectrum, that solves the higher dimensional equations of motion, is called *consistent* although the term is now generally applied to any solution of the full theory that is independent of the internal coordinates, regardless of whether or not the solution arose from a truncation of a harmonic expansion. In contrast to the Kaluza-Klein approach the effective theory obtained may not include all of the lightest modes and the number of preserved fields will not generally be related to the topological properties of the internal manifold.

There is some overlap between these categories and examples exist of Kaluza-Klein truncations which are consistent. An interesting example is compactification on a torus for which a truncation to the lightest harmonics (Fourier modes) yields an effective lower dimensional theory. The resulting effective theory solves the higher dimensional equations of motion and is an example of both procedures. As a counter-example, compactification on a Calabi-Yau followed by truncation to the zero modes is not consistent in this sense.

The consistency of a reduction can often be understood group theoretically.¹ For example, in the case of a reduction on a torus discussed above, the momenta of the compactified fields along the internal directions give rise to conserved charges in the D -dimensional theory. Truncating to the zero modes in this case leads to keeping only the singlets, therefore there can be no chance of interactions generating the modes that have been truncated out. This truncation is therefore consistent.

Various generalisations of the toroidal reduction which are consistent (i.e. do not depend on the internal coordinates and solve the higher-dimensional equations of motion) have been systematically studied [2–8]. These include the Scherk-Schwarz reductions [7, 8] and reductions with cohomological fluxes [5, 6] which yield effective theories with scalar potentials and non-abelian gauge symmetries. These reductions may be thought of as massive deformations of the toroidal solution.

In this paper we consider Scherk-Schwarz reductions of IIB supergravity and F-Theory that admit a geometric interpretation² as a compactification on a manifold $\mathcal{X} = G/\Gamma$.

¹Coset reductions appear to be an exception. There is, as yet, no systematic understanding of the consistency of such reductions, despite the numerous examples that exist [2–4].

²We describe the Scherk-Schwarz solution as a compactification on \mathcal{X} in the sense that the reduction

Here G is a, possibly non-compact, d -dimensional group manifold and $\Gamma \subset G$ is a discrete subgroup acting from the left such that \mathcal{X} is compact. The group manifold G has isometry group $G_L \times G_R$ (the action of the group on itself from the left and right respectively) which is broken to G_R by the discrete quotient (see [9] for a detailed discussion), leaving an effective theory with G_R gauge symmetry. In order for the reduced fields to be globally defined on \mathcal{X} the Scherk-Schwarz ansatz requires the reduction ansatz to be invariant under the rigid action of G_L [9, 10]. The spaces \mathcal{X} are known as twisted tori in the literature. Generally \mathcal{X} will have no relation to a torus fibration so the terminology is misleading. Henceforth $\mathcal{X} = G/\Gamma$ shall be referred to as an *identified group manifold*.³ Examples of identified-group manifolds that are topologically twisted torus fibrations were given in [9].

This construction and its generalisation to include matter with flux is reviewed in the following section. Section 3 presents a study of the flux compactification of IIB Supergravity on identified group manifolds with particular focus on the gauge symmetry and its breaking. The Kaluza-Klein reduction has a clear geometric interpretation by construction. The consistent reductions may not always be easily identified with the truncation of a Kaluza-Klein reduction on some manifold and it is interesting to consider the higher-dimensional origin of these reductions. In contrast to the reduction via harmonic analysis, many solutions generally cannot be interpreted in terms of a geometric compactification. Many examples now exist [9–16] of solutions of the higher dimensional theory, depending only on a set of macroscopic coordinates, for which the internal space cannot be understood in terms of classical Riemannian geometry. In section 4 we consider S-duality twisted reductions of IIB Supergravity. These reductions do not arise from a geometric reduction of IIB Supergravity but may be interpreted as a compactification of F-Theory on an identified group manifold.

2. Flux compactifications on identified group manifolds

In this section the Scherk-Schwarz reduction on an identified group manifold \mathcal{X} is reviewed. The coordinates of the higher-dimensional space-time are $x^M = (x^\mu, y^i)$ where y^i ($i = 1, 2, \dots, d$) are coordinates on \mathcal{X} and x^μ ($\mu = d + 1, d + 2, \dots, d + D$) are the coordinates on the non-compact spacetime. The most general Einstein frame reduction ansatz invariant under rigid G_L is

$$d\hat{s}^2 = e^{2\alpha\varphi} ds_D^2 + e^{2\beta\varphi} g_{mn} \nu^m \nu^n \tag{2.1}$$

where the one-forms

$$\nu^m = \sigma^m - A^m \tag{2.2}$$

include the Kaluza-Klein gauge fields A_μ^m , which have two-form field strength

$$F^m = dA^m + \frac{1}{2} f_{np}{}^m A^n \wedge A^p \tag{2.3}$$

ansatz is constructed from the set of globally defined one-forms on \mathcal{X} .

³Suggested by C. Hull. Other names one might consider are ‘Cocompact Orbifold’ or ‘Coset’, but these names are misleading as Γ is discrete and acts freely on G .

and α, β in (2.1) are the constants

$$\alpha = - \left(\frac{d}{2(D-2)(D+d-2)} \right)^{\frac{1}{2}} \quad \beta = - \frac{\alpha(D-2)}{d} \quad (2.4)$$

The left-invariant one-forms $\sigma^m = \sigma^m_i(y)dy^i$, where $m = 1, 2, \dots, d$, satisfy the structure equation

$$d\sigma^m + \frac{1}{2}f_{np}^m \sigma^n \wedge \sigma^p = 0 \quad (2.5)$$

which ensures that all y^i -dependence drops out of the reduced theory. The integrability condition $d^2\sigma^m = 0$ gives the algebraic constraint $f_{[mn}^q f_p]q^t = 0$, and the invariance of the internal measure under G_R requires that G_R be unimodular⁴ ($f_{mn}^n = 0$).

The ν^m define a covariant basis for the reduction in which the one forms transform under the local right action G_R , generated by the globally defined left-invariant vector fields $Z_m = \sigma_m^i \partial_i$ as

$$\delta_Z(\omega)y^i = \omega^m \sigma_m^i \quad \delta_Z(\omega)\nu^m = -\nu^n f_{np}^m \omega^p \quad \delta_Z(\omega)A^m = -d\omega^m - A^n f_{np}^m \omega^p \quad (2.6)$$

Dimensional reduction gives rise to a metric $g_{\mu\nu}(x)$, d Kaluza-Klein one-form gauge fields $A_\mu^m(x)$, and $d(d+1)/2$ scalars $\varphi(x)$ and $g_{mn}(x)$, where $g_{mn}(x)$ is a positive definite symmetric matrix with unit determinant. This ansatz (2.1) is invariant under rigid G_L transformations, and under local G_R transformations in which the parameters depend on x^μ and the A^m transform as gauge fields, while the scalar fields $g_{mn}(x)$ transform in the bi-adjoint.

The $D+d$ -dimensional Einstein-Hilbert Lagrangian, reduced on a d -dimensional identified group manifold \mathcal{X}_d , gives the effective theory

$$\begin{aligned} \mathcal{L}_D = & R * 1 - \frac{1}{2} * d\varphi \wedge d\varphi - \frac{1}{2} g^{mp} g^{nq} * Dg_{mn} \wedge Dg_{pq} - \frac{1}{2} e^{2(\beta-\alpha)\varphi} g_{mn} * F^m \wedge F^n \\ & - \frac{1}{4} e^{2(\alpha-\beta)\varphi} (g_{mn} g^{pq} g^{ts} f_{pt}^m f_{qs}^n + 2g^{mn} f_{qm}^p f_{pn}^q) * 1 \end{aligned} \quad (2.7)$$

where

$$Dg_{mn} = dg_{mn} + g_{mp} f_{nq}^p A^q + g_{np} f_{mq}^p A^q \quad (2.8)$$

is a G_R -covariant derivative.

Theories of interest to us will also include antisymmetric tensor fields. In the ansatz of [8], the internal components $T_{ij\dots k}$ of a tensor field $\widehat{T}_{MN\dots P}$ in the reduced theory are taken to have y dependence given only by the frame fields

$$T_{ij\dots k}(x, y) = T_{mn\dots p}(x) \sigma_i^m \sigma_j^n \dots \sigma_k^p \quad (2.9)$$

defining scalar fields $T_{mn\dots p}(x)$ in the reduced theory, so that for example the internal metric takes the form $g_{ij}(x, y) = g_{mn}(x) \sigma_i^m \sigma_j^n$. As an example, consider the antisymmetric two-form potential, which we write in the (G_R -covariant) ν^m basis as

$$\widehat{B}_{(2)} = B_{(2)} + B_{(1)m} \wedge \nu^m + \frac{1}{2} B_{(0)mn} \nu^m \wedge \nu^n + \varpi_{(2)} \quad (2.10)$$

⁴Relaxing the unimodular condition still allows the reduction of the equations of motion, but not the Lagrangian.

where a left-invariant flux has been introduced

$$\mathcal{K} = \frac{1}{6} K_{mnp} \sigma^m \wedge \sigma^n \wedge \sigma^p \quad (2.11)$$

where $\mathcal{K} = d\varpi_{(2)}$. The flux is closed (which requires the algebraic constraint $K_{[mn|p} f_{|qt]}^p = 0$) but generally not exact so that $\varpi_{(2)}$ is defined only locally. Defining the algebraic operator \mathcal{Q} , which acts on the space of antisymmetric tensors $K_{m_1 m_2 \dots m_p} = K_{[m_1 m_2 \dots m_p]}$, such that

$$\mathcal{Q} : K_{m_1 m_2 \dots m_p} \rightarrow (\mathcal{Q}K)_{m_1 m_2 \dots m_{p+1}} = f_{[m_1 m_2}{}^n K_{m_3 m_4 \dots m_{p+1}]n} \quad (2.12)$$

The condition $K_{[mn|p} f_{|qt]}^p = 0$ may be written as $(\mathcal{Q}K)_{mnpq} = 0$, or $K_{mnp} \in \text{Ker } \mathcal{Q}$. The condition $f_{[mn}{}^q f_{p]q}{}^t = 0$ means that $\mathcal{Q}^2 = 0$ and we may define the algebraic cohomology $H(\mathcal{Q}) = \text{Ker } \mathcal{Q} / \text{Im } \mathcal{Q}$. It was shown in [9] that a flux of the form $K_{mnp} = (\mathcal{Q}\eta)_{mnp}$ for some $\eta_{mn} = -\eta_{nm}$ can be removed by a field redefinition of B and is therefore trivial. Therefore the fluxes of interest are those in $H(\mathcal{Q})$. For field strengths with more complicated Bianchi identities $dH \neq 0$, then $K_{[mn|p} f_{|qt]}^p \neq 0$ and the statement of algebraic cohomology must be suitably modified.

The field strength $\widehat{H}_{(3)} = d\widehat{B}_{(2)}$ is invariant under the transformation $\delta_X(\lambda) \widehat{B}_{(2)} = d\widehat{\lambda}_{(1)}$ where

$$\widehat{\lambda}_{(1)} = \lambda_{(1)} + \lambda_{(0)m} \nu^m \quad (2.13)$$

and $\lambda_{(1)} = \lambda_\mu dx^\mu$. The antisymmetric tensor symmetry and the G_R symmetry of \mathcal{X} gives the infinitesimal transformations

$$\begin{aligned} \delta B_{(2)} &= d\lambda_{(1)} + \frac{1}{2} K_{mnp} \omega^p A^m \wedge A^n \\ \delta B_{(1)m} &= D\lambda_{(0)m} + B_{(1)n} f_{mp}{}^n \omega^p - K_{mnp} \omega^p A^n \\ \delta B_{(0)mn} &= f_{mn}{}^p \lambda_{(0)p} + 2B_{(0)[m|p} f_{|n]q}{}^p \omega^q + K_{mnp} \omega^p \end{aligned} \quad (2.14)$$

where $D\lambda_{(0)m} = d\lambda_{(0)m} + f_{mn}{}^p \lambda_{(0)p} A^n$. Combining these variations with that of the graviphoton $\delta A^m = -D\omega^m$, these infinitesimals generate the Lie algebroid

$$\begin{aligned} [\delta_Z(\tilde{\omega}), \delta_Z(\omega)] &= \delta_Z(f_{np}{}^m \omega^n \tilde{\omega}^p) - \delta_X(K_{mnp} \omega^n \tilde{\omega}^p) - \delta_W(K_{mnp} \omega^n \tilde{\omega}^p A^m) \\ [\delta_X(\lambda), \delta_Z(\omega)] &= -\delta_X(\lambda_m f_{np}{}^m \omega^p) \\ [\delta_X(\tilde{\lambda}), \delta_X(\lambda)] &= 0 \end{aligned} \quad (2.15)$$

where $\delta_Z(\omega) = \omega^m Z_m$, $\delta_X(\lambda) = \lambda_{(0)m} X^m$ and $\delta_W(\lambda) = \lambda_\mu W^\mu$. As argued in [9, 10], such field dependence is characteristic of theories in which we require field strengths to have Chern-Simons terms in order to be gauge invariant and such Chern-Simons terms are generated naturally by dimensional reduction. The algebra may be viewed as a Lie algebra bundle over the non-compact D -dimensional spacetime, i.e. at each point x_o on the base, the graviphoton is constant along the fibre and the algebroid reduces to a Lie algebra with structure constants $f_{mn}{}^p$, K_{mnp} and $K_{mnp} A_\mu^p(x_o)$.

The algebroid (2.15) has Lie subalgebra [22]

$$\begin{aligned} [Z_m, Z_n] &= -f_{mn}{}^p Z_p + K_{mnp} X^p \\ [X^m, Z_n] &= f_{np}{}^m X^p \\ [X^m, X^n] &= 0 \end{aligned} \quad (2.16)$$

In the next section we shall consider a generalisation of this reduction to the full bosonic sector of the IIB supergravity.

3. Compactifications of IIB supergravity

We consider here the reduction of the bosonic sector of IIB supergravity on identified group manifolds with flux. The general reduction of the Fermi sector and Supersymmetry breaking will be considered elsewhere.⁵ An important property of the IIB theory is the self-duality of the Ramond-Ramond five-form field strength.

$$\widehat{G}_{(5)} = *\widehat{G}_{(5)} \tag{3.1}$$

Such a constraint cannot naturally be encoded in a Lagrangian formalism and it must be separately imposed on the equations of motion. The approach will be to treat $\widehat{G}_{(5)}$ and $*\widehat{G}_{(5)}$ as independent fields in the Lagrangian and impose the self-duality constraint after the dimensional reduction. We shall only be interested in the general structure of the reduced theory, in particular the gauge symmetries, so the issue of self duality will not play a significant role.

The bosonic sector of the ten-dimensional IIB Lagrangian, written in a manifestly SL(2) invariant form is

$$\mathcal{L}_{IIB} = \widehat{\mathcal{R}}*1 + \frac{1}{4}tr \left(*d\widehat{\mathcal{K}} \wedge d\widehat{\mathcal{K}}^{-1} \right) - \frac{1}{2}\widehat{\mathcal{K}}_{ab}*\widehat{\mathcal{H}}_{(3)}^a \wedge \widehat{\mathcal{H}}_{(3)}^b - \frac{1}{4}*\widehat{G}_{(5)} \wedge \widehat{G}_{(5)} - \frac{1}{4}\epsilon_{ab}\widehat{\mathcal{C}}_{(4)} \wedge \widehat{\mathcal{H}}_{(3)}^a \wedge \widehat{\mathcal{H}}_{(3)}^b \tag{3.2}$$

Where the 3-form field strengths $\widehat{\mathcal{H}}_{(3)}^a$ transform as a doublet under SL(2), the self-dual five-form $\widehat{G}_{(5)}$ as a singlet and the axio-dilaton τ (written above in terms of the scalars $\widehat{\mathcal{K}}$) in a fractionally linear way

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2) \tag{3.3}$$

The trace in (3.2) is taken over the SL(2) indices $a = 1, 2$. The field strengths and scalars $\widehat{\mathcal{K}}$ are defined;

$$\widehat{G}_{(5)} = d\widehat{\mathcal{C}}_{(4)} + \frac{1}{2}\epsilon_{ab}\widehat{\mathcal{B}}_{(2)}^a \wedge \widehat{\mathcal{H}}_{(3)}^b \quad \widehat{\mathcal{H}}_{(3)}^a = d\widehat{\mathcal{B}}_{(2)}^a = \begin{pmatrix} db_{(2)} \\ dc_{(2)} \end{pmatrix} \quad \widehat{\mathcal{K}} = e^\phi \begin{pmatrix} 1 & C_{(0)} \\ C_{(0)} & |\tau|^2 \end{pmatrix} \tag{3.4}$$

where

$$\epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \tau = C_{(0)} + ie^{-\phi} \tag{3.5}$$

The scalar sector consists of a dilaton ϕ and a Ramond-Ramond zero-form $C_{(0)}$ which parameterise the coset manifold $\text{SL}(2; \mathbb{R})/\text{SO}(2) \simeq \text{SU}(1, 1)/\text{U}(1)$. $\widehat{\mathcal{C}}_{(4)}$ and $c_{(2)}$ are p-form fields arising from the massless Ramond-Ramond sector of the Type IIB String spectrum

⁵See [17] for a recent discussion of $\mathcal{N} = 1$ vacua in the case where \mathcal{X} is a six-dimensional Nil or Solve-manifold.

and $b_{(2)}$ is the Kalb-Ramond potential. The ten dimensional Equations of motion derived from the Lagrangian (3.2) are

$$\begin{aligned} d * \widehat{G}_{(5)} &= \frac{1}{2} \epsilon_{ab} \widehat{H}_{(3)}^a \wedge \widehat{H}_{(3)}^b \\ d * \widehat{\mathcal{K}}_{ab} \widehat{H}_{(3)}^a &= \epsilon_{ab} \widehat{H}_{(3)}^b \wedge \widehat{G}_{(5)} \\ d * d\widehat{\mathcal{K}}^{ab} &= * \widehat{H}_{(3)}^a \wedge \widehat{H}_{(3)}^b \end{aligned} \quad (3.6)$$

and the Bianchi identities are

$$d\widehat{G}_{(5)} - \frac{1}{2} \epsilon_{ab} \widehat{\mathcal{H}}^a \wedge \widehat{\mathcal{H}}^b = 0 \quad d\widehat{\mathcal{H}}^a = 0 \quad (3.7)$$

which are consistent with the self-duality constraint (3.1). The action of S-duality on these fields is

$$\widehat{\mathcal{K}} \rightarrow S^t \widehat{\mathcal{K}} S \quad \widehat{\mathcal{B}}_{(2)} \rightarrow S^{-1} \widehat{\mathcal{B}}_{(2)} \quad (3.8)$$

and ϵ_{ab} is invariant

$$\epsilon \rightarrow S^t \epsilon S = \epsilon \quad (3.9)$$

where $S \in \text{SL}(2; \mathbb{Z}) \subset \text{SL}(2)$.

3.1 Inclusion of fluxes

The flux ansatz for the two form is a generalisation of that described in [9, 22], which transforms covariantly under S-duality, mixing the Neveu-Schwarz and Ramond two form fluxes. A left-invariant flux, $M_{(3)}^a$, is included in the two form reduction via the ansatz

$$\widehat{\mathcal{B}}_{(2)}^a = \widehat{B}_{(2)}^a + \varpi_{(2)}^a \quad \widehat{\mathcal{H}}_{(3)}^a = \widehat{H}_{(3)}^a + M_{(3)}^a \quad (3.10)$$

where $\widehat{H}_{(3)}^a = d\widehat{B}_{(2)}^a$ and

$$d\varpi_{(2)}^a = M_{(3)}^a = \frac{1}{6} M_{mnp}{}^a \sigma^m \wedge \sigma^n \wedge \sigma^p \quad (3.11)$$

$M_{mnp}{}^a$ are constant, $\text{SL}(2)$ valued antisymmetric coefficients. Introducing flux on the $\widehat{G}_{(5)}$ field strength is not straightforward due to the Chern-Simons term in $\widehat{G}_{(5)}$. Such terms threaten the consistency of the truncation as they introduce bare flux potential terms $\varpi_{(2)}^a$ which have explicit y dependence. In [22] it was demonstrated that (for the Heterotic string), by a careful choice of flux, the consistency of the truncation may be maintained even for theories with such Chern-Simons terms. These techniques can be generalised to higher degree forms and applied to the $\widehat{\mathcal{C}}_{(4)}$ potential of the IIB theory to give the flux ansatz

$$\widehat{\mathcal{C}}_{(4)} = \widehat{S}_{(4)} - \frac{1}{2} \epsilon_{ab} \varpi_{(2)}^a \wedge \widehat{B}_{(2)}^b + \varpi_{(4)} \quad (3.12)$$

where

$$d\varpi_{(4)} = -\frac{1}{2} \epsilon_{ab} \varpi_{(2)}^a \wedge M_{(3)}^b + \mathcal{K}_{(5)} \quad (3.13)$$

$\mathcal{K}_{(5)}$ is the left-invariant five-form flux

$$\mathcal{K}_{(5)} = \frac{1}{120} K_{mnpqt} \sigma^m \wedge \sigma^n \wedge \sigma^p \wedge \sigma^q \wedge \sigma^t \quad (3.14)$$

The first term on the right hand side in (3.13) is required to cancel any y -dependence in the five form field strength that may arise due to the flux on $\widehat{\mathcal{B}}_{(2)}^a$ in the Chern-Simons term of $\widehat{\mathcal{G}}_{(5)}$. The reduction ansatz for the five-form field strength (3.4) is

$$\widehat{\mathcal{G}}_{(5)} = d\widehat{\mathcal{S}}_{(4)} + \frac{1}{2} \epsilon_{ab} \left(\widehat{\mathcal{B}}_{(2)}^a \wedge \widehat{\mathcal{H}}_{(3)}^b - 2M_{(3)}^a \wedge \widehat{\mathcal{B}}_{(2)}^b \right) + \mathcal{K}_{(5)} \quad (3.15)$$

It will be important, especially when considering the symmetry transformations, to distinguish between those parts of the potentials with flux included implicitly, denoted by the calligraphic script $\widehat{\mathcal{B}}_{(2)}^a$ and $\widehat{\mathcal{C}}_{(4)}$, and those without, $\widehat{\mathcal{B}}_{(2)}^a$ and $\widehat{\mathcal{S}}_{(4)}$. The requirement that the fluxes do not alter the Bianchi identities (3.7) requires that the fluxes satisfy

$$d \left(\frac{1}{6} M_{mnp}{}^a \sigma^m \wedge \sigma^n \wedge \sigma^p \right) = 0 \quad d \left(-\frac{1}{2} \epsilon_{ab} \varpi_{(2)}^a \wedge M_{(3)}^b + \mathcal{K}_{(5)} \right) = 0 \quad (3.16)$$

which impose the algebraic conditions

$$\begin{aligned} M_{[mn]t}{}^a f_{[pq]}{}^t &= 0 \\ 2\epsilon_{ab} M_{[mnp]}{}^a M_{[qts]}{}^b + 3K_{[mnpq]l} f_{[ts]}{}^l &= 0 \end{aligned} \quad (3.17)$$

In addition to the condition $f_{[mn}{}^q f_{p]}{}^t = 0$.

3.2 Flux compactification on identified group manifolds

The Chern-Simons term of the ten-dimensional IIB Lagrangian has an explicit dependence on the potential of the fluxes $\varpi_{(2)}^a$ and $\varpi_{(4)}$, entering through $\widehat{\mathcal{C}}_{(4)}$. It is the fluxes $M_{(3)}^a$ and $\mathcal{K}_{(5)}$ that are globally defined, not the potentials so one might worry that the Lagrangian is not well defined and the reduction not consistent. However variation of the Lagrangian with respect to the potentials $\widehat{\mathcal{S}}_{(4)}$ and $\widehat{\mathcal{B}}_{(2)}^a$ still yield the correct, well defined, equations of motion (3.6). One way to proceed would be to disregard the Lagrangian (3.2) and reduce the equations of motion (3.6) directly.

The fact that the physics depends only on the fluxes $M_{(3)}^a$ and $\mathcal{K}_{(5)}$ and not the potentials $\varpi_{(2)}^a$ and $\varpi_{(4)}$ is due to the gauge invariance of the theory under antisymmetric tensor transformations. However, even though the equations of motion are manifestly invariant under the tensor transformations, the Lagrangian is not. Consider the Chern-Simons form contribution to the action

$$S_{CS} = \frac{1}{4} \int_M \epsilon_{ab} \widehat{\mathcal{C}}_{(4)} \wedge \widehat{\mathcal{H}}_{(3)}^a \wedge \widehat{\mathcal{H}}_{(3)}^b \quad (3.18)$$

under the antisymmetric tensor transformation $\delta_X \widehat{\mathcal{C}} = d\widehat{\Lambda}$ the action transforms as

$$\begin{aligned} \delta_X S_{CS} &= \frac{1}{4} \int_M \epsilon_{ab} d\widehat{\Lambda} \wedge \widehat{\mathcal{H}}_{(3)}^a \wedge \widehat{\mathcal{H}}_{(3)}^b \\ &= \frac{1}{4} \int_{\partial M} \epsilon_{ab} \widehat{\Lambda} \wedge \widehat{\mathcal{H}}_{(3)}^a \wedge \widehat{\mathcal{H}}_{(3)}^b \end{aligned} \quad (3.19)$$

which is zero if either the ten-dimensional spacetime has no boundary or $\widehat{\Lambda}$ vanishes on the boundary. The problem arises when one considers large gauge transformations. The fact that S_{CS} is not manifestly invariant under large gauge transformations is related to the appearance of the bare flux potentials $\varpi_{(2)}^a$ and $\varpi_{(4)}$. This issue, in addition to the self-duality constraint provides good motivation to consider the reductions of the equations of motion directly. However there are some issues, such as moduli fixing, in which it can be helpful to have an explicit reduction of the scalar potential.

The left-invariant Scherk-Schwarz reduction ansatz are

$$\begin{aligned}\widehat{S}_{(4)} &= S_{(4)} + S_{(3)m} \wedge \nu^m + \frac{1}{2} S_{(2)mn} \wedge \nu^m \wedge \nu^n + \frac{1}{6} S_{(1)mnp} \wedge \nu^m \wedge \nu^n \wedge \nu^p \\ &\quad + \frac{1}{24} S_{(0)mnpq} \nu^m \wedge \nu^n \wedge \nu^p \wedge \nu^q \\ \widehat{G}_{(5)} &= G_{(5)} + G_{(4)m} \wedge \nu^m + \frac{1}{2} G_{(3)mn} \wedge \nu^m \wedge \nu^n + \frac{1}{6} G_{(2)mnp} \wedge \nu^m \wedge \nu^n \wedge \nu^p \\ &\quad + \frac{1}{24} G_{(1)mnpq} \wedge \nu^m \wedge \nu^n \wedge \nu^p \wedge \nu^q + \frac{1}{120} G_{(0)mnpqt} \nu^m \wedge \nu^n \wedge \nu^p \wedge \nu^q \wedge \nu^t\end{aligned}\tag{3.20}$$

where $\widehat{G}_{(5)}$ is defined by (3.4) For the two-form we define

$$\begin{aligned}\widehat{B}_{(2)}^a &= B_{(2)}^a + B_{(1)m}^a \wedge \nu^m + \frac{1}{2} B_{(0)mn}^a \nu^m \wedge \nu^n \\ \widehat{H}_{(3)}^a &= H_{(3)}^a + H_{(2)m}^a \wedge \nu^m + \frac{1}{2} H_{(1)mn}^a \wedge \nu^m \wedge \nu^n + \frac{1}{6} H_{(0)mnp}^a \nu^m \wedge \nu^n \wedge \nu^p\end{aligned}\tag{3.21}$$

and similarly for the three form field strength with flux $\widehat{\mathcal{H}}_{(3)}^a$

$$\widehat{\mathcal{H}}_{(3)}^a = \mathcal{H}_{(3)}^a + \mathcal{H}_{(2)m}^a \wedge \nu^m + \frac{1}{2} \mathcal{H}_{(1)mn}^a \wedge \nu^m \wedge \nu^n + \frac{1}{6} \mathcal{H}_{(0)mnp}^a \nu^m \wedge \nu^n \wedge \nu^p\tag{3.22}$$

The reduced field strengths and Bianchi identities are given in appendix A. The reduced theory has scalar potential

$$\begin{aligned}V &= -\frac{1}{2} e^{2(\beta-\alpha)\varphi} (g_{mn} g^{pq} g^{ts} f_{pt}^m f_{qs}^n + 2g^{mn} f_{qm}^p f_{pn}^q) \\ &\quad - \frac{1}{4} e^{-10(\beta-\alpha)\varphi} g^{mn} g^{pq} g^{ts} g^{lk} g^{ij} G_{(0)mptli} G_{(0)nqskj} \\ &\quad - \frac{1}{2} e^{-6(\beta-\alpha)\varphi} g^{mn} g^{pq} g^{ts} \mathcal{K}_{ab} \mathcal{H}_{(0)mpt}^a \mathcal{H}_{(0)nqs}^b\end{aligned}\tag{3.23}$$

3.3 Gauge symmetry

In this section the gauge symmetries of the IIB theory reduced on an identified group manifold \mathcal{X} with flux described in the previous sections are investigated. The presence of the Chern-Simons term in $\widehat{G}_{(5)}$ leads to a gauge algebra with a far more complicated structure than seen in (2.15).

3.3.1 Three form anti-symmetric tensor transformations

The antisymmetric tensor transformation $\widehat{S}_{(4)} \rightarrow \widehat{S}_{(4)} + d\widehat{\Lambda}_{(3)}$ leaves the field strength $\widehat{G}_{(5)}$ invariant and is generated by the parameters

$$\widehat{\Lambda}_{(3)} = \Lambda_{(3)} + \Lambda_{(2)m} \wedge \nu^m + \frac{1}{2} \Lambda_{(1)mn} \wedge \nu^m \wedge \nu^n + \frac{1}{6} \Lambda_{(0)mnp} \nu^m \wedge \nu^n \wedge \nu^p\tag{3.24}$$

Gauge transformations with respect to each component of $\widehat{\Lambda}_{(3)}$ are generated by $\delta_X(\Lambda_{(3)})$, $\delta_X(\Lambda_{(2)m})$, $\delta_X(\Lambda_{(1)mn})$ and $\delta_X(\Lambda_{(0)mnp})$. The generators of this transformation are defined as $\delta_X(\widehat{\Lambda})$ and their action on the reduced potential is

$$\begin{aligned}
 \delta_X(\widehat{\Lambda})S_{(4)} &= d\Lambda_{(3)} - \Lambda_{(2)m} \wedge F^m \\
 \delta_X(\widehat{\Lambda})S_{(3)m} &= D\Lambda_{(2)m} + \Lambda_{(1)mn} \wedge F^n \\
 \delta_X(\widehat{\Lambda})S_{(2)mn} &= -\Lambda_{(2)p} f_{mn}^p + D\Lambda_{(1)mn} - \Lambda_{(0)mnp} F^p \\
 \delta_X(\widehat{\Lambda})S_{(1)mnp} &= O_{mnp}^{qt} \Lambda_{(1)qt} + D\Lambda_{(0)mnp} \\
 \delta_X(\widehat{\Lambda})S_{(0)mnpq} &= -O_{mnpq}^{tsl} \Lambda_{(0)tsl}
 \end{aligned} \tag{3.25}$$

where constants O_{mnp}^{qt} and O_{mnpq}^{tsl} are defined as

$$\begin{aligned}
 O_{mnp}^{qt} &= 3\delta_{[m}^q f_{np]}^t \\
 O_{mnpq}^{tsl} &= 6\delta_{[m}^t \delta_n^s f_{pq]}^l
 \end{aligned} \tag{3.26}$$

3.3.2 One-form anti-symmetric tensor transformations

Consider the symmetry generated by the gauge transformation

$$\delta_Y(\widehat{\lambda})\widehat{B}_{(2)}^a = \delta_Y(\widehat{\lambda})\widehat{B}_{(2)}^a = d\widehat{\lambda}_{(1)}^a \tag{3.27}$$

The three form field strength $\widehat{\mathcal{H}}_{(3)}^a$ is manifestly invariant under this transformation, but invariance of the five form $\widehat{G}_{(5)}$ requires a compensating transformation from $\widehat{S}_{(4)}$. The self-dual five form field strength is

$$\widehat{G}_{(5)} = d\widehat{S}_{(4)} + \frac{1}{2}\epsilon_{ab} \left(\widehat{B}_{(2)}^a \wedge d\widehat{B}_{(2)}^b - 2M_{(3)}^a \wedge \widehat{B}_{(2)}^b \right) + \mathcal{K}_{(5)} \tag{3.28}$$

We define the effect of the infinitesimal transformation $\delta_Y(\widehat{\lambda})$ on $\widehat{S}_{(4)}$ as that which ensures; $\delta_Y(\widehat{\lambda})\widehat{G}_{(5)} = 0$, i.e.

$$\delta_Y(\widehat{\lambda})\widehat{G}_{(5)} = d \left(\delta_Y(\widehat{\lambda})\widehat{S}_{(4)} \right) + \frac{1}{2}\epsilon_{ab} \left(d\widehat{\lambda}_{(1)}^a \wedge d\widehat{B}_{(2)}^b - 2M_{(3)}^a \wedge d\widehat{\lambda}_{(1)}^b \right) = 0 \tag{3.29}$$

Integrating this equation gives an expression for the gauge transformation of $\widehat{S}_{(4)}$;

$$\delta_Y(\widehat{\lambda})\widehat{S}_{(4)} = -\frac{1}{2}\epsilon_{ab} \left(d\widehat{\lambda}_{(1)}^a \wedge \widehat{B}_{(2)}^b + 2\widehat{\lambda}_{(1)}^a \wedge M_{(3)}^b \right) + d\Lambda_{(3)} \tag{3.30}$$

where $\Lambda_{(3)}$ is an arbitrary 3-form, which we shall set to zero. The gauge parameter and its exterior derivative are

$$\begin{aligned}
 \widehat{\lambda}_{(1)}^a &= \lambda_{(1)}^a + \lambda_{(0)m}^a \nu^m \\
 d\widehat{\lambda}_{(1)}^a &= d\lambda_{(1)}^a - \lambda_{(0)m}^a F^m + D\lambda_{(0)m}^a \wedge \nu^m - \frac{1}{2}\lambda_{(0)p}^a f_{mn}^p \nu^m \wedge \nu^n
 \end{aligned} \tag{3.31}$$

Substituting (3.31) in (3.30), the one-form gauge transformations are

$$\begin{aligned}
 \delta_Y(\widehat{\lambda})S_{(4)} &= -\frac{1}{2}\epsilon_{ab} \left(d\lambda_{(1)}^a - \lambda_{(0)m}^a F^m \right) \wedge B_{(2)}^b - \frac{1}{6}\epsilon_{ab}\lambda_{(1)}^a M_{mnp}^b \wedge A^m \wedge A^n \wedge A^p \\
 \delta_Y(\widehat{\lambda})S_{(3)m} &= -\frac{1}{2}\epsilon_{ab} \left(d\lambda_{(1)}^a - \lambda_{(0)n}^a F^n \right) \wedge B_{(1)m}^b - \frac{1}{2}\epsilon_{ab}\lambda_{(1)}^a M_{mnp}^b \wedge A^n \wedge A^p \\
 &\quad - \frac{1}{2}\epsilon_{ab} D\lambda_{(0)m}^a \wedge B_{(2)}^b - \frac{1}{6}\epsilon_{ab}\lambda_{(0)m}^a M_{npq}^b A^n \wedge A^p \wedge A^q
 \end{aligned}$$

$$\begin{aligned}
 \delta_Y(\widehat{\lambda})S_{(2)mn} &= -\frac{1}{2}\epsilon_{ab}\left(d\lambda_{(1)}^a - \lambda_{(0)p}^a F^p\right)B_{(0)mn}^b - \epsilon_{ab}\lambda_{(1)}^a \wedge M_{mnp}{}^b A^p \\
 &\quad + \frac{1}{2}\epsilon_{ab}\lambda_{(0)p}^a f_{mn}{}^p B_{(2)}^b - \epsilon_{ab}D\lambda_{(0)[m}^a \wedge B_{(1)n]}^b - \epsilon_{ab}\lambda_{(0)[m}^a M_{n]pq}{}^b A^p \wedge A^q \\
 \delta_Y(\widehat{\lambda})S_{(1)mnp} &= -\epsilon_{ab}\lambda_{(1)}^a M_{mnp}{}^b - \frac{3}{2}\epsilon_{ab}D\lambda_{(0)[m}^a B_{(0)np]}^b - 3\epsilon_{ab}\lambda_{(0)[m}^a M_{np]q}{}^b A^q \\
 &\quad + \frac{3}{2}\epsilon_{ab}\lambda_{(0)q}^a f_{[mn}{}^q B_{(1)p]}^b \\
 \delta_Y(\widehat{\lambda})S_{(0)mnpq} &= 6\epsilon_{ab}\lambda_{(0)t}^a f_{[mn}{}^t B_{(0)pq]}^b - 4\epsilon_{ab}\lambda_{(0)[m}^a M_{npq]}{}^b
 \end{aligned} \tag{3.32}$$

and for the two form

$$\begin{aligned}
 \delta_Y(\widehat{\lambda})B_{(2)}^a &= d\lambda_{(1)}^a - \lambda_{(0)m}^a F^m \\
 \delta_Y(\widehat{\lambda})B_{(1)m}^a &= D\lambda_{(0)m}^a \\
 \delta_Y(\widehat{\lambda})B_{(0)mn}^a &= -\lambda_{(0)p}^a f_{mn}{}^p
 \end{aligned} \tag{3.33}$$

3.3.3 Right action of the group manifold

$\mathcal{X} = G/\Gamma$ inherits the right action of the group G_R on G . The calculation of how the reduced fields transform under G_R is somewhat involved and only the results are given here. Details of the calculation may be found in appendix B. The right action gives

$$\begin{aligned}
 \delta_Z(\omega^m)S_{(4)} &= \frac{1}{4}\epsilon_{ab}M_{mnp}{}^a \omega^p B_{(2)}^b \wedge A^m \wedge A^n - \frac{1}{24}K_{mnpqt}\omega^t A^m \wedge A^n \wedge A^p \wedge A^q \\
 \delta_Z(\omega^m)S_{(3)m} &= S_{(3)n}f_{mp}{}^n \omega^p + \frac{1}{4}\epsilon_{ab}M_{npq}{}^a \omega^q A^n \wedge A^p \wedge B_{(1)m}^b \\
 &\quad + \frac{1}{2}\epsilon_{ab}M_{mnp}{}^a \omega^p A^n \wedge B_{(2)}^b + \frac{1}{6}K_{mnpqt}\omega^t A^n \wedge A^p \wedge A^q \\
 \delta_Z(\omega^m)S_{(2)mn} &= 2S_{(2)[m|p}f_{|n]q}{}^p \omega^q + \frac{1}{4}\epsilon_{ab}M_{pqt}{}^a \omega^t A^p \wedge A^q B_{(0)mn}^b + \epsilon_{ab}M_{mpq}{}^a \omega^p A^q \wedge B_{(1)n}^b \\
 &\quad + \frac{1}{2}\epsilon_{ab}M_{mnp}{}^a \omega^p B_{(2)}^b - \frac{1}{2}K_{mnpqt}\omega^t A^p \wedge A^q \\
 \delta_Z(\omega^m)S_{(1)mnp} &= 3S_{(1)[mn|q}f_{|p]t}{}^q \omega^t + \frac{3}{2}\epsilon_{ab}M_{mnq}{}^a \omega^q B_{(1)p}^b - \frac{3}{2}\epsilon_{ab}M_{mqt}{}^a \omega^t B_{(0)np}^b A^q \\
 &\quad + K_{mnpqt}\omega^t A^q \\
 \delta_Z(\omega^m)S_{(0)mnpq} &= 4S_{(0)[mnp|t}f_{|q]s}{}^t \omega^s + 3\epsilon_{ab}M_{[mn|t}{}^a B_{(0)|pq]}^b \omega^t - K_{mnpqt}\omega^t
 \end{aligned} \tag{3.34}$$

3.3.4 Gauge algebra

Using the results of the previous sections, the full gauge algebra of the compactified IIB theory is

$$\begin{aligned}
 [\delta_Z(\bar{\omega}), \delta_Z(\omega)] &= \delta_Z(f_{np}{}^m \omega^n \bar{\omega}^p) - \delta_X(K_{mnpqt}\omega^q \bar{\omega}^t) - \delta_X(K_{mnpqt}\omega^q \bar{\omega}^t A^p) \\
 &\quad - \delta_X\left(\frac{1}{2}K_{mnpqt}\omega^q \bar{\omega}^t A^n \wedge A^p\right) \\
 &\quad - \delta_X\left(\frac{1}{6}K_{mnpqt}\omega^q \bar{\omega}^t A^m \wedge A^n \wedge A^p\right) \\
 &\quad - \delta_Y(M_{mnp}{}^a \omega^n \bar{\omega}^p) - \delta_Y(M_{mnp}{}^a \omega^n \bar{\omega}^p A^m)
 \end{aligned}$$

$$\begin{aligned}
 [\delta_X(\Lambda_{(2)m}), \delta_Z(\omega^m)] &= \delta_X(\Lambda_{(2)n} f_{mp}{}^n \omega^p) \\
 [\delta_X(\Lambda_{(1)mn}), \delta_Z(\omega^q)] &= \delta_X(\Lambda_{(1)mp} f_{nq}{}^p \omega^q) - \delta_X(\Lambda_{(1)np} f_{mq}{}^p \omega^q) \\
 [\delta_X(\Lambda_{(0)mnp}), \delta_Z(\omega^m)] &= \delta_X(\Lambda_{(0)mnp} f_{pt}{}^q \omega^t) + \delta_X(\Lambda_{(0)npq} f_{mt}{}^q \omega^t) + \delta_X(\Lambda_{(0)pmq} f_{nt}{}^q \omega^t) \\
 [\delta_Y(\lambda_{(1)}^a), \delta_Z(\omega^m)] &= -\delta_X(\epsilon_{ab} M_{mnp}{}^b \omega^p \lambda_{(1)}^b) - \delta_X(\epsilon_{ab} M_{mnp}{}^a \omega^p \lambda_{(1)}^b \wedge A^n) \\
 &\quad - \delta_X\left(\frac{1}{2} \epsilon_{ab} M_{mnp}{}^a \omega^p \lambda_{(1)}^b \wedge A^m \wedge A^n\right) \\
 [\delta_Y(\lambda_{(0)m}^a), \delta_Z(\omega^m)] &= \delta_Y(\lambda_{(0)n}^a f_{mp}{}^n \omega^p) - \delta_X(3\epsilon_{ab} \lambda_{(0)m}^a M_{npq}{}^b \omega^q)
 \end{aligned}$$

All other commutators vanish. This gauge algebra contains a Lie algebra subgroup. A naive guess for the Lie algebra is

$$\begin{aligned}
 [Z_m, Z_n] &= -f_{mn}{}^p Z_p - M_{mnp}{}^a Y_a{}^p - K_{mnpqt} X^{pqt} \\
 [X^{mnp}, Z_q] &= 3f_{qt}{}^{[m} X^{np]t} \\
 [Y_a{}^m, Z_n] &= f_{np}{}^m Y_a{}^p - 3\epsilon_{ab} M_{npq}{}^b X^{mpq}
 \end{aligned} \tag{3.35}$$

with all other commutators vanishing. As in the eleven dimensional supergravity case [10], the Jacobi identity for this algebra fails to close and a truncation of the set of generators must be considered. This is a consequence of the reducibility of the gauge transformations. It will be shown in the examples of section 3.4 that the irreducible gauge transformations correspond to the irreducible representations of the gauge group. Consider for example the slightly simpler case where $M_{mnp}{}^a = 0$ then algebra (3.35) reduces to

$$\begin{aligned}
 [Z_m, Z_n] &= -f_{mn}{}^p Z_p - K_{mnpqt} X^{pqt} \\
 [X^{mnp}, Z_q] &= f_{qt}{}^m X^{npt} + f_{qt}{}^p X^{mnt} + f_{qt}{}^n X^{pmt}
 \end{aligned} \tag{3.36}$$

The triple commutator for Z_m is

$$\begin{aligned}
 [[Z_m, Z_n], Z_p] + [[Z_n, Z_p], Z_m] + [[Z_p, Z_m], Z_n] &= 3K_{mnp[q|t} f_{|sl]}{}^t X^{qsl} \\
 &= K_{mnpjt} O_{qsl}^{jt} X^{qsl} \neq 0
 \end{aligned} \tag{3.37}$$

where the constants O_{mnp}^q and Π_{qt}^{mnp} are

$$\begin{aligned}
 O_{mnp}^{qt} &= 3\delta_{[m}^q f_{np]}{}^t \\
 \Pi_{qt}^{mnp} &= \frac{1}{2} \delta_{[q}^{[m} f_{t}{}^{np]}
 \end{aligned} \tag{3.38}$$

We see that the commutators (3.36) do not satisfy the Jacobi identity and is therefore not a Lie algebra. The apparent non-associativity of (3.36) may be understood by considering the following example. For simplicity consider the case where the group G the identified group manifold is constructed from is chosen to be semi-simple. It is useful to decompose

the generator X^{mnp} as⁶

$$X^{mnp} = \tilde{X}^{mnp} + \Pi_{qt}^{mnp} \tilde{X}^{qt} \quad (3.40)$$

where $O_{mnp}^{qt} \tilde{X}^{mnp} = 0$ and $f_{np}{}^m \tilde{X}^{np} = 0$, such that $O_{mnp}^{qt} X^{mnp} = \tilde{X}^{qt}$. The algebra

$$\begin{aligned} [Z_m, Z_n] &= -f_{mn}{}^p Z_p - K_{mnpqt} \tilde{X}^{pqt} \\ [\tilde{X}^{mnp}, Z_q] &= 3f_{qt}{}^{[m} \tilde{X}^{np]t} \end{aligned} \quad (3.41)$$

satisfies the Jacobi identity and is a Lie subalgebra of the algebroid (3.35). Of course, the full symmetry algebra (3.35) satisfies the Jacobi identity, but is not a Lie algebra. It will be shown in section 3.4.2 that the action of \tilde{X}^{mn} on all potentials is trivial so that the non-trivial action of the antisymmetric tensor transformation is generated by \tilde{X}^{mnp} alone. Adding in the three form flux $M_{mnp}{}^a$ and allowing the twisted torus to be non-semi-simple, the symmetry algebra contains the Lie algebra

$$\begin{aligned} [Z_m, Z_n] &= -f_{mn}{}^p Z_p - M_{mnp}{}^a Y_a{}^p - K_{mnpqt} \tilde{X}^{pqt} \\ [\tilde{X}^{mnp}, Z_q] &= 3f_{qt}{}^{[m} \tilde{X}^{np]t} \\ [Y_a{}^m, Z_n] &= f_{np}{}^m Y_a{}^p - 3\epsilon_{ab} M_{npq}{}^b \tilde{X}^{mpq} \end{aligned} \quad (3.42)$$

where \tilde{X}^{mnp} satisfies

$$O_{mnp}^{qt} \tilde{X}^{mnp} = 0 \quad M_{mnp}{}^a \tilde{X}^{mnp} = 0 \quad (3.43)$$

These two constraints are required in order for the Jacobi identity to be satisfied. The first is simply a generalisation of the decomposition in (3.40) for non-semi-simple G . To understand the second constraint $M_{mnp}{}^a \tilde{X}^{mnp} = 0$ consider the transformation of the scalar field

$$\delta_X(\hat{\Lambda}) S_{(0)mnpq} = -O_{mnpq}^{tsl} \Lambda_{(0)tsl} \quad (3.44)$$

and now using the decomposition

$$\Lambda_{(0)mnp} = \tilde{\Lambda}_{(0)mnp} + M_{mnp}{}^a \tilde{\Lambda}_{(0)a} \quad (3.45)$$

Under the anti-symmetric tensor transformation generated by the parameter $M_{mnp}{}^a \tilde{\Lambda}_{(0)a}$ the scalars $S_{(0)mnpq}$ are invariant

$$\begin{aligned} \delta_X(\hat{\Lambda}) S_{(0)mnpq} &= -O_{mnpq}^{tsl} M_{tsl}{}^a \tilde{\Lambda}_{(0)a} \\ &= -6f_{[mn}{}^l M_{pq]l}{}^a \tilde{\Lambda}_{(0)a} = 0 \end{aligned} \quad (3.46)$$

where the last equality is a consequence of the flux integrability condition (3.17). Therefore the symmetry with parameters $\tilde{\Lambda}_{(0)a}$ and $\tilde{\Lambda}_{(0)mn}$ leave the scalar field $S_{(0)mnpq}$ invariant and drop out of the symmetry algebra altogether, in accordance with the Jacobi identity above.

⁶Consider the further decomposition, $\tilde{X}^{qt} \rightarrow \tilde{X}^{qt} + f_s{}^{qt} \tilde{X}^s$, where $f_{mn}{}^p \tilde{X}^{mn} = 0$. The action of Π_{qt}^{mnp} projects out the \tilde{X}^s contribution, since

$$\Pi_{qt}^{mnp} \tilde{X}^{qt} \rightarrow \Pi_{qt}^{mnp} \tilde{X}^{qt} + \frac{1}{2} f_t{}^{[np} f_s{}^{m]t} \tilde{X}^s = \Pi_{qt}^{mnp} \tilde{X}^{qt} \quad (3.39)$$

Therefore, for our purposes, this second decomposition need not be explicitly stated, except to note that $f_{mn}{}^p \tilde{X}^{mn} = 0$.

The naive algebra (3.35), with generators $T_M = (Z_m, X^{mnp}, Y_a^m)$, is not a Lie algebra but, as we have seen, there exists a truncated set of generators $\tilde{T}_M = (Z_m, \tilde{X}^{mnp}, Y_a^m)$ which do generate a Lie algebra. Similar observations from various flux compactifications of ten- and eleven-dimensional supergravity have been made by many authors and it has been shown that free differential algebras [18, 19] have an important role to play in understanding the algebraic structure of the reduced theories. Indeed, the algebroid (3.35) contains a free differential algebra which can be identified as the algebra (3.35). In particular, in [20, 21] it was pointed out that, in order for the free differential algebra that arises from such reductions to give a Lie algebra, certain additional constraints must be placed on the structure constants such that the algebra is associative. Following [20, 21], one would be tempted to impose the constraint $K_{mnp[q|t}f_{|sl]}^t = K_{mnpjt}O_{qsl}^{jt} = 0$ on the structure constants in (3.35). It is possible to find physical configurations in which this condition arises naturally; however, from the above discussion and the examples presented in section 3.4 we see that such a condition is not necessary here. The component \tilde{X}^{mn} of the generator which is responsible for the apparent failure of the Jacobi identity does not act on any physical field in the theory and so the action of the generators, restricted to the physical fields, ensures that the Jacobi identity is satisfied on the space of fields. In other words, the generators T_M and \tilde{T}_M have the same action on the fields of the theory. Further examples, as applied to compactifications of eleven-dimensional supergravity, may be found in [10].

3.4 Examples and symmetry breaking

In this section the symmetry breaking down to a linearly realised subgroup that is generic for any solution shall be discussed. For vacua with vanishing scalar expectation value, this is the complete breaking, but for non-trivial scalar expectation values there will be further breaking through the standard Higgs mechanism. The examples considered in 3.4.2 and 3.4.3 are the two extreme cases, one where the fluxes vanish and the reduction is the standard Scherk-Schwarz one and the second in which the structure constants f_{mn}^p vanish but the fluxes do not. In the following sub-section symmetry breaking in the gravity sector, i.e. the sector described by (2.7) is discussed. This symmetry breaking is generic for any such theory.

3.4.1 Symmetry breaking in the gravity sector

The breaking of the local G_R symmetry by a choice of vacuum is easy to analyse in the Scherk-Schwarz reduction. The metric transforms in the bi-adjoint representation

$$\delta_Z(\omega)g_{mn} = g_{mp}f_{nq}^p\omega^q + g_{np}f_{mq}^p\omega^q \tag{3.47}$$

These transformations are only isometries for the cases where the metric is invariant $\delta_Z(\omega)g_{mn} = 0$, i.e. the frame directions $\sigma^{\bar{q}}$ for which

$$g_{mp}f_{n\bar{q}}^p + g_{np}f_{m\bar{q}}^p = 0 \tag{3.48}$$

are isometric and the generators $Z_{\bar{q}}$ generate isometries of the metric g_{mn} mediated by the gauge bosons $A^{\bar{q}}$. All directions $\sigma^{\bar{q}}$ for which

$$g_{mp}f_{n\bar{q}}^p + g_{np}f_{m\bar{q}}^p \neq 0 \tag{3.49}$$

correspond to symmetries $Z_{\hat{q}}$ which are broken by the choice of vacuum. The gauge bosons $A^{\hat{q}}$ of these broken symmetries have mass-like terms in the Lagrangian arising from the kinetic term $*Dg_{mn} \wedge Dg_{pq}$ of (2.7)

$$\mathcal{L}_D = -(g^{mn} g^{pq} f_{mt}{}^p f_{ns}{}^q - f_{tm}{}^n f_{sn}{}^m) * A^{\hat{t}} \wedge A^{\hat{s}} + \dots \quad (3.50)$$

If the metric g_{mn} acquires a vacuum expectation value \bar{g}_{mn} , then this becomes a mass term for those graviphotons which are *not* associated to isometries of the frozen metric \bar{g}_{mn} , through the Higgs mechanism

$$\mathcal{L}_D = -\frac{1}{2} M_{\hat{t}\hat{s}} * A^{\hat{t}} \wedge A^{\hat{s}} + \dots \quad (3.51)$$

where the mass matrix $M_{\hat{t}\hat{s}}$ is given by

$$M_{\hat{t}\hat{s}} = 2(\bar{g}^{mn} \bar{g}^{pq} f_{mt}{}^p f_{ns}{}^q - f_{tm}{}^n f_{sn}{}^m) \quad (3.52)$$

A vacuum in which the scalars have the expectation value $\bar{g}_{mn} = \eta_{mn}$, the (bi-invariant) Cartan-Killing metric (3.56) will be invariant under G_R while any other expectation value \bar{g}_{mn} will break the gauge symmetry to the subgroup preserving \bar{g}_{mn} .

3.4.2 Reduction on an identified group manifold with semi-simple right action

The reduction on $\mathcal{X} = G/\Gamma$ where G is any semi-simple group is considered. All fluxes are taken to be zero and therefore $\widehat{\mathcal{C}}_{(4)} = \widehat{S}_{(4)}$ and $\widehat{\mathcal{B}}_{(2)}^a = \widehat{B}_{(2)}^a$. The gauge algebra in this case is a Lie algebra

$$\begin{aligned} [Z_m, Z_n] &= -f_{mn}{}^p Z_p \\ [X^{mnp}, Z_q] &= 3f_{qt}{}^{[m} X^{np]t} \\ [Y_a{}^m, Z_n] &= f_{np}{}^m Y_a{}^p \end{aligned} \quad (3.53)$$

with all other commutators vanishing. This algebra generates the group $G_R \times \mathrm{U}(1)^q$ where $q = d + \binom{d}{3}$ which is broken to the linearly realised subgroup $G_R \times \mathrm{U}(1)^q$ by any vacuum of the theory as will be shown.

For the purposes of this section, the one-form antisymmetric tensor transformations are chosen to be

$$\delta_Y(\widehat{\lambda}) S_{(4)} = -\frac{1}{2} \epsilon_{ab} \widehat{\lambda}_{(1)}^a \wedge \widehat{H}_{(3)}^b \quad (3.54)$$

where $\widehat{\mathcal{H}}_{(3)}^a = \widehat{H}_{(3)}^a$. This choice is related to that in (3.30) by a choice of the arbitrary parameter $\Lambda_{(3)}$. The non-linear gauge transformations are

$$\begin{aligned} \delta B_{(0)mn}^a &= -\lambda_{(0)p}^a f_{mn}{}^p + \dots \\ \delta S_{(2)mn} &= -\Lambda_{(2)p} f_{mn}{}^p + \dots \\ \delta S_{(1)mnp} &= 3\Lambda_{(1)[m|t} f_{np]}{}^t + \dots \\ \delta S_{(0)mnpq} &= -6\Lambda_{(0)[mn|t} f_{pq]}{}^t + \dots \end{aligned} \quad (3.55)$$

where $+\dots$ denote linear terms. On a semi-simple identified group manifold the Cartan-Killing metric

$$\eta_{mn} = \frac{1}{2} f_{mp}^q f_{nq}^p \quad (3.56)$$

is non-degenerate and invertible. The inverse metric η^{mn} may be used to define $f_m^{np} = \eta^{nq} f_{mq}^p$. These constants may be viewed as maps $f_{mn}^p \xi_p \rightarrow \xi_{mn}$ and $f_p^{mn} \xi_{mn} \rightarrow \xi_p$ for some antisymmetric $\xi_{mn\dots} = \xi_{[mn\dots]}$

$$f : \mathbb{R}^d \rightarrow \mathbb{R}^{\binom{d}{2}} \quad f^{-1} : \mathbb{R}^{\binom{d}{2}} \rightarrow \mathbb{R}^d \quad (3.57)$$

and satisfy $f_{mp}^q f_q^{np} = 2\delta_m^n$. It will also be useful to recall the definition of O_{mnp}^{qt} and also to define the constant Π_{qt}^{mnp} as

$$\begin{aligned} O_{mnp}^{qt} &= 3\delta_{[m}^q f_{np]}^t \\ \Pi_{qt}^{mnp} &= \frac{1}{2} \delta_q^{[m} f_t^{np]} \end{aligned} \quad (3.58)$$

These constants may be viewed as maps; $\xi_{mnp} \rightarrow O_{mnp}^{[qt]} \xi_{qt}$ and $\xi_{mn} \rightarrow \Pi_{[mn]}^{pqt} \xi_{pqt}$, or more abstractly as

$$O : \mathbb{R}^{\binom{d}{2}} \rightarrow \mathbb{R}^{\binom{d}{3}} \quad \Pi : \mathbb{R}^{\binom{d}{3}} \rightarrow \mathbb{R}^{\binom{d}{2}} \quad (3.59)$$

Note that these maps are not inverses of each other but satisfy the identity

$$\Pi_{[mn]}^{tsl} O_{tsl}^{[pq]} = \delta_{mn}^{pq} - \frac{1}{2} f_{mn}^t f_t^{pq} \quad (3.60)$$

We also define constants $O_{mnpq}^{ts,l}$ and $\Pi_{ts,l}^{mnpq}$ as

$$\begin{aligned} O_{mnpq}^{ts,l} &= 6\delta_{[mn}^{ts} f_{pq]}^l \\ \Pi_{ts,l}^{mnpq} &= \frac{1}{2} \delta_{ts}^{[mn} f_l^{pq]} \end{aligned} \quad (3.61)$$

which may be thought of as maps defined by $O_{mnpq}^{[ts,l]} \xi_{tsl} = \xi_{mnpq}$ and $\Pi_{[ts,l]}^{mnpq} \xi^{tsl} = \xi^{mnpq}$

$$O : \mathbb{R}^{\binom{d}{3}} \rightarrow \mathbb{R}^{\binom{d}{4}} \quad \Pi : \mathbb{R}^{\binom{d}{4}} \rightarrow \mathbb{R}^{\binom{d}{3}} \quad (3.62)$$

and satisfy the relationship

$$\Pi_{[mn,p]}^{ijkl} O_{ijkl}^{[qt,s]} = \delta_{mnp}^{qts} - O_{mnp}^{[ij]} \Pi_{[ij]}^{qts} \quad (3.63)$$

A number of other useful identities that these constants obey are collected in appendix A of [10]. The following identities are also useful

$$\begin{aligned} D^2 \xi_m &= f_{mn}^p \xi_p F^n \\ D^2 \xi_{mn} &= (O_{mnp}^{qt} - \delta_p^q f_{mn}^t) \xi_{qt} F^p \end{aligned} \quad (3.64)$$

The potentials $S_{(0)mnpq}$, $S_{(1)mnp}$, $S_{(2)mn}$ and $S_{(3)m}$ take values in the $\binom{d}{4}$, $\binom{d}{3}$, $\binom{d}{2}$ and d dimensional representations of $\text{SL}(d; \mathbb{R})$ respectively. In order to understand the possible

field redefinitions that are required to remove the non-linear group actions, the gauge parameters must be decomposed into these representations. The $\Lambda_{(2)m}$ can only take values in the d representation but we may decompose $\Lambda_{(1)mn}$ and $\Lambda_{(0)mnp}$ as

$$\Lambda_{(1)mn} = \tilde{\Lambda}_{(1)mn} + f_{mn}{}^p \tilde{\Lambda}_{(1)p} \quad (3.65)$$

and

$$\Lambda_{(0)mnp} = \tilde{\Lambda}_{(0)mnp} + O_{mnp}{}^{qt} \tilde{\Lambda}_{(0)qt} \quad (3.66)$$

where $\tilde{\Lambda}_{(1)mn}$ and $\tilde{\Lambda}_{(0)mnp}$ satisfy⁷

$$f_p{}^{mn} \tilde{\Lambda}_{(1)mn} = 0 \quad \Pi_{qt}{}^{mnp} \tilde{\Lambda}_{(0)mnp} = 0 \quad (3.67)$$

In terms of these parameters the transformations (3.25) are

$$\begin{aligned} \delta_X(\Lambda)S_{(4)} &= d\Lambda_{(3)} - \Lambda_{(2)m} \wedge F^m \\ \delta_X(\Lambda)S_{(3)m} &= D\left(\Lambda_{(2)m} + D\tilde{\Lambda}_{(1)m} + \tilde{\Lambda}_{(0)mn}F^n\right) - \left(D\tilde{\Lambda}_{(1)mn} - \tilde{\Lambda}_{(1)mn}\right) \wedge F^n \\ \delta_X(\Lambda)S_{(2)mn} &= D\left(\tilde{\Lambda}_{(1)mn} - D\tilde{\Lambda}_{(0)mn}\right) - \tilde{\Lambda}_{(0)mnp}F^p \\ &\quad - f_{mn}{}^p \left(\Lambda_{(2)p} + D\tilde{\Lambda}_{(1)p} + \tilde{\Lambda}_{(0)pq}F^q\right) \\ \delta_X(\Lambda)S_{(1)mnp} &= -D\tilde{\Lambda}_{(0)mnp} + O_{mnp}{}^{qt} \left(D\tilde{\Lambda}_{(1)qt} - \tilde{\Lambda}_{(1)qt}\right) \\ \delta_X(\Lambda)S_{(0)mnpq} &= -O_{mnpq}{}^{tsl} \tilde{\Lambda}_{(0)tsl} \end{aligned} \quad (3.68)$$

The goldstone bosons of the broken symmetries (3.55) are given by

$$\begin{aligned} \chi_{(0)m}^a &= \frac{1}{2} f_m{}^{np} B_{(0)np}^a \\ \chi_{(0)mnp} &= \Pi_{mn,p}{}^{qtsl} S_{(0)qtsl} \\ \chi_{(1)mn} &= \Pi_{mn}{}^{pqt} S_{(1)pqt} \\ \chi_{(2)m} &= \frac{1}{2} f_m{}^{np} S_{(2)np} \end{aligned} \quad (3.69)$$

These Goldstone bosons transform as

$$\begin{aligned} \delta_Y(\lambda^a)\chi_{(0)m}^a &= \lambda_{(0)m}^a \\ \delta_X(\Lambda)\chi_{(0)mnp} &= -\tilde{\Lambda}_{(0)mnp} \\ \delta_X(\Lambda)\chi_{(1)mn} &= D\tilde{\Lambda}_{(0)mn} - \tilde{\Lambda}_{(1)mn} \\ \delta_X(\Lambda)\chi_{(2)m} &= \Lambda_{(2)m} + D\tilde{\Lambda}_{(1)m} + \tilde{\Lambda}_{(0)mn}F^n \end{aligned} \quad (3.70)$$

⁷As discussed in a footnote to section 3.3.4, a further decomposition $\tilde{\Lambda}_{(0)mn} = \bar{\Lambda}_{(0)mn} + f_{mn}{}^p \bar{\Lambda}_{(0)p}$ (where $f_p{}^{mn} \bar{\Lambda}_{(0)mn} = 0$) is redundant since the parameter $\bar{\Lambda}_{(0)p}$ is projected out in (3.66) due to the identity $O_{mnp}{}^{qt} f_{qt}{}^s = 0$. We may therefore neglect $\bar{\Lambda}_{(0)p}$ and enforce the constraint $f_p{}^{mn} \tilde{\Lambda}_{(0)mn} = 0$ without loss of generality.

One may therefore define the following $\delta_Y(\lambda_{(0)}^a)$ and $\delta_X(\Lambda_{(0)mnp}, \Lambda_{(1)mn}, \Lambda_{(2)m})$ -invariant potentials

$$\begin{aligned}
 \check{S}_{(4)} &= S_{(4)} - \chi_{(2)m} \wedge F^m \\
 \check{S}_{(3)m} &= S_{(3)m} - \frac{1}{2} \epsilon_{ab} \chi_{(0)m}^a H_{(3)}^b - D\chi_{(2)m} + \chi_{(1)mn} \wedge F^n \\
 \check{S}_{(2)mn} &= S_{(2)mn} + \epsilon_{ab} \chi_{(0)m}^a H_{(2)n}^b - D\chi_{(1)mn} + f_{mn}{}^p \chi_{(2)p} + \chi_{(0)mnp} F^p \\
 \check{S}_{(1)mnp} &= S_{(1)mnp} - \frac{3}{2} \epsilon_{ab} \chi_{(0)m}^a H_{(1)np}^b - D\chi_{(0)mnp} + O_{mnp}^{qt} \chi_{(1)qt} \\
 \check{S}_{(0)mnpq} &= S_{(0)mnpq} + 2\epsilon_{ab} \chi_{(0)m}^a H_{(0)npq}^b - O_{mnpq}^{tsl} \chi_{(0)tsl}
 \end{aligned} \tag{3.71}$$

similarly for the B^a -fields

$$\begin{aligned}
 \check{B}_{(2)}^a &= B_{(2)}^a + \chi_{(0)m}^a \wedge F^m \\
 \check{B}_{(1)m}^a &= B_{(1)m}^a - D\chi_{(0)m}^a \\
 \check{B}_{(0)mn}^a &= B_{(0)mn}^a + f_{mn}{}^p \chi_{(0)p}^a
 \end{aligned} \tag{3.72}$$

These field redefinitions take the form of infinitesimal gauge transformations (even though the Goldstone field need not be small) so the form of the field strengths are not changed by the redefinition except to replace the potentials (B^a, S) by the (\check{B}^a, \check{S}) defined above. The gauge algebra is reduced to

$$[Z_m, Z_n] = -f_{mn}{}^p Z_p \tag{3.73}$$

with all other commutators vanishing, generating the group $G_R \times \text{U}(1)^q$ as claimed.

3.4.3 Flux reduction on a torus

If $f_{mn}{}^p = 0$, then the group G_R is abelian and the internal manifold (after discrete identifications to compactify, if necessary) is a torus and one may take $G_R = \text{U}(1)^d$. With flux $\mathcal{K}_{(5)}$ and $M_{(3)}^a$, the gauge algebra (3.35) has the Lie sub-algebra

$$\begin{aligned}
 [Z_m, Z_n] &= -M_{mnp}{}^a Y_a^p - K_{mnpqt} X^{pqt} \\
 [Y_a^m, Z_n] &= -3\epsilon_{ab} M_{npq}{}^b X^{mpq}
 \end{aligned} \tag{3.74}$$

with all other commutators vanishing.

Non-linear realisation of the right action. As a warm up, consider the two-form sector in isolation. Viewing $\alpha^m \rightarrow \alpha^m M_{mnp}{}^a$ as a map

$$M : \mathbb{R}^d \rightarrow \mathbb{R}^{d(d-1)} \tag{3.75}$$

the internal index m can be split into (m', \bar{m}) , so that \bar{m} labels the $(d-d')$ dimensional kernel of the map M , and m' labels the cokernel, so that

$$M_{mn\bar{p}}{}^a = 0 \quad M_{m'n'p'}{}^a \neq 0 \tag{3.76}$$

Then the transformation of the $B_{(0)}^a$ scalars is

$$\delta B_{(0)n'p'}^a = \omega^{m'} M_{m'n'p'}^a, \quad \delta B_{(0)m\bar{n}}^a = 0 \quad (3.77)$$

The transformations generated by $Z_{m'}$ with parameters $\omega^{m'}$ are spontaneously broken by any vacuum of the theory. For a vacuum $(\bar{g}, \bar{\phi}, \bar{K})$ the $A^{m'}$ fields have mass term in the Lagrangian

$$\mathcal{L}_D = -\frac{1}{2} e^{-4\beta\bar{\phi}} \bar{g}^{mn} \bar{g}^{pq} \bar{K}_{ab} M_{mpt'}^a M_{nqs'}^b * A^{t'} \wedge A^{s'} + \dots \quad (3.78)$$

The $2d$ dimensional gauge group is broken to the $(2d - d')$ dimensional abelian subgroup $U(1)^{2d-d'}$ generated by $Z_{\bar{m}}$ and X^m with parameters $\omega^{\bar{m}}$ and $\lambda_{(0)m}^m$ respectively. Let $\tilde{M}^{m'n'p'}_a$ be any constants satisfying $\tilde{M}^{m'n'p'}_a M_{n'p'q'}^b = \delta^{m'}_{q'} \delta_a^b$. Then the Goldstone fields $\chi_{(0)}^{m'}$ are defined by

$$\chi_{(0)}^{m'} = \tilde{M}^{m'n'p'}_a B_{(0)n'p'}^a \quad (3.79)$$

transforming as a shift

$$\delta B_{(0)\bar{M}}^a = 0 \quad \delta \chi_{(0)}^{m'} = \omega^{m'} \quad (3.80)$$

The remaining scalars are invariant, $\delta B_{(0)m\bar{n}}^a = 0$. The massive graviphotons are defined as $\check{A}^{m'} = A^{m'} + d\chi_{(0)}^{m'}$ and are singlets of the gauge transformations.

Non-linear realisation of the one-form antisymmetric tensor transformation.

If the four form potentials are now introduced, the one-form antisymmetric tensor transformations appear as shift symmetries

$$\begin{aligned} \delta B_{(0)mnp}^a &= M_{mnp}^a \omega^p + \dots \\ \delta S_{(1)mnp} &= -\epsilon_{ab} \lambda_{(1)}^a M_{mnp}^b + \dots \\ \delta S_{(0)mnpq} &= -4\epsilon_{ab} \lambda_{(0)m}^a M_{npq}^b + K_{mnpqt} \omega^t + \dots \end{aligned} \quad (3.81)$$

Therefore, even on a flat torus, the presence of flux will break some of the anti-symmetric tensor transformations generated by Y_a^m . The gauge bosons of symmetries with parameters $\lambda_{(0)m}^a$ and $\lambda_{(1)}^a$ are $B_{(2)}^a$ and $B_{(1)m}^a$ respectively and some of these potentials become massive by the Higgs mechanism. As in the flat torus case above, one may define constants \tilde{M}^{mnp}_a such that $M_{mnp}^b \tilde{M}^{mnp}_a = \delta_a^b$, such that the goldstone one-form $\chi_{(1)a}$ defined as

$$\chi_{(1)a} = S_{(1)mnp} \tilde{M}^{mnp}_a \quad (3.82)$$

which transforms as $\delta \chi_{(1)a} = \epsilon_{ab} \lambda_{(1)}^b + \dots$. The $S_{(0)mnpq}$ transformation in (3.81) may be written as

$$\begin{aligned} \delta S_{(0)mnpq} &= -4\epsilon_{ab} \lambda_{(0)m}^a M_{npq}^b + K_{mnpqt} \omega^t + \dots \\ &= -R_{mnpq}^t \lambda_{(0)t}^a + K_{mnpqt} \omega^t + \dots \end{aligned} \quad (3.83)$$

where it is useful to define the constant $R_{mnpq}{}^t{}_a = 4\epsilon_{ab}\delta^t{}_{[m}M_{npq]}^b$. This transformation may be written as

$$\begin{aligned}\delta S_{(0)\Sigma} &= \left(-\lambda_{(0)t}^a \omega^t\right) \begin{pmatrix} R_{mnpq}{}^t{}_a \\ K_{mnpqt} \end{pmatrix} + \dots \\ &= \alpha_{(0)M} t^M{}_{\Sigma} + \dots\end{aligned}\tag{3.84}$$

where the index $M = 1, 2 \dots 3d$ so that

$$\alpha_{(0)M} = \left(-\lambda_{(0)m}^1 - \lambda_{(0)m}^2 \omega^m\right)\tag{3.85}$$

and the index $\Sigma = [mnpq] = 1, 2 \dots \binom{d}{4}$. Treating $t^M{}_{\Sigma}$ as the map

$$t : \mathbb{R}^{3d} \rightarrow \mathbb{R}^{\binom{d}{4}}\tag{3.86}$$

the index M may be split into $M = (M', \bar{M})$ where M' and \bar{M} label the cokernel and kernel of the map t respectively. A basis may then be chosen such that the constant tensor $t^M{}_{\Sigma}$ is written as

$$t^M{}_{\Sigma} = \begin{pmatrix} t^{M'}{}_{\Sigma'} & 0 \\ 0 & 0 \end{pmatrix}\tag{3.87}$$

The choice of basis is such that there exists an inverse $\tilde{t}^{\Sigma'}{}_{M'}$ where $t^{M'}{}_{\Sigma'} \tilde{t}^{\Sigma'}{}_{N'} = \delta^{M'}{}_{N'}$ and $\tilde{t}^{\Sigma'}{}_{M'} t^{M'}{}_{\Lambda'} = \delta^{\Sigma'}{}_{\Lambda'}$. The Goldstone boson for the symmetry with parameter $\alpha_{(0)M'}$ is

$$\chi_{(0)M'} = \tilde{t}^{\Sigma'}{}_{M'} S_{(0)\Sigma'}\tag{3.88}$$

It is useful to combine the generators Z_m and Y_a^m into the doublet

$$T^M = \begin{pmatrix} Y_a^m \\ Z_m \end{pmatrix}\tag{3.89}$$

so that $\delta = \alpha_M T^M$. Those symmetries generated by $T^{\bar{M}}$ (with parameter $\alpha_{\bar{M}}$) are preserved whilst those generated by $T^{M'}$ (with parameter $\alpha_{M'}$) have non-linear realisations and are always broken by a choice of vacuum of the theory. Gauge singlet fields may be defined

$$\begin{aligned}\check{S}_{(0)\Sigma'} &= S_{(0)\Sigma'} - \chi_{(0)M'} t^{M'}{}_{\Sigma'} \\ \check{S}_{(1)mnp} &= S_{(1)mnp} + \chi_{(1)a} M_{mnp}{}^a \\ \check{B}_{(0)mn}^a &= B_{(0)mn} - \chi_{(0)}{}^p M_{mnp}{}^a\end{aligned}\tag{3.90}$$

Both the $\mathcal{H}_{(1)mn}^a$ and $G_{(1)mnpq}$ field strengths contribute to the graviphoton mass term where $G_{(1)mnpq} = K_{mnpqt} A^t + \dots$ and $\mathcal{H}_{(1)mn}^a = M_{mnp}{}^a A^p + \dots$. For a given vacuum expectation value of the scalars \bar{g} , \bar{K} and $\bar{\varphi}$, the mass-like term in the Lagrangian due to these field strengths may be written as

$$\mathcal{L}_D = -\frac{1}{2} \mathcal{M}_{AB} * \mathcal{A}_{(1)}^A \wedge \mathcal{A}_{(1)}^B + \dots\tag{3.91}$$

where the components of the mass matrix are

$$\begin{aligned}
 \mathcal{M}_{mn} &= e^{4\beta\bar{\varphi}} \bar{g}^{mt} \bar{g}^{ns} \bar{g}^{pl} \bar{g}^{qk} \left(e^{4(\alpha-\beta)\bar{\varphi}} K_{mnpqi} K_{tslkj} + \frac{1}{d} \bar{g}_{lq} \bar{g}_{pk} \bar{\mathcal{K}}_{ab} M_{mni}{}^a M_{tsj}{}^b \right) \\
 \mathcal{M}_m{}^n &= -2e^{4\beta\bar{\varphi}} \bar{g}^{mt} \bar{g}^{ns} \bar{g}^{pl} \bar{g}^{qk} e^{4(\alpha-\beta)\bar{\varphi}} \epsilon_{ab} M_{tsl}{}^a K_{mnpqi} \\
 \mathcal{M}^{mn} &= 4e^{4\beta\bar{\varphi}} \bar{g}^{mt} \bar{g}^{ns} \bar{g}^{pl} \bar{g}^{qk} \epsilon_{ac} \epsilon_{bd} M_{mnp}{}^c M_{qts}{}^d
 \end{aligned} \tag{3.92}$$

and

$$\mathcal{A}_{(1)}^A = \begin{pmatrix} A^m \\ B_{(1)m}^a \end{pmatrix} \tag{3.93}$$

For a given vacuum expectation value of the scalar fields the diagonalisation of the matrix \mathcal{M}_{AB} gives the (mass)² spectrum of the $\mathcal{A}_{(1)}^A$ potentials. In general, with non-trivial internal geometry some of the $S_{(1)mnp}$ gauge fields will become massive, corresponding to the breaking of symmetries generated by \tilde{X}^{mnp} . Upon symmetry breaking by the choice of some vacuum $(\bar{g}, \bar{\varphi}, \bar{\mathcal{K}})$, the general effective theory will contain massive one-forms and massless gauge bosons that are linear combinations of the $B_{(1)m}^a$, A^m and $S_{(1)mnp}$.

4. Compactifications with S-duality twists and F-theory

Consider a $D + d + 1$ dimensional field theory coupled to gravity. The theory is reduced on a d -dimensional torus T^d , with real coordinates $z^a \sim z^a + 1$ where $a = 1, 2 \dots d$. This produces a theory in $D + 1$ dimensions with scalar fields that include those in the coset $GL(d, \mathbb{R})/SO(d)$ arising from the torus moduli. Truncating to the z^a independent zero mode sector, this theory has a global symmetry U that contains the $GL(d, \mathbb{R})$ arising from diffeomorphisms of the d -dimensional torus. In the full Kaluza-Klein theory this is broken to the $GL(d, \mathbb{Z})$ that acts as large diffeomorphisms on the d -dimensional torus similarly, in string theory U is broken to the discrete U-duality subgroup $U(\mathbb{Z})$. The action of U on fields ψ of the reduced theory in some representation of U is denoted as $\psi \rightarrow \gamma[\psi]$.

The duality twist reductions of this theory describes reduction on a further circle with periodic coordinate $y \sim y + 1$, twisting the fields over the circle by an element of U using the ansatz [11, 23, 24, 15]

$$\psi(x^\mu, y) = \gamma_y[\psi(x^\mu)] \tag{4.1}$$

where x^μ are the D non-compact spacetime coordinates. Consistency of the reduction, in the sense described in the introduction, requires the reduced theory to be independent of y , which is achieved by choosing the form of γ to be

$$\gamma(y) = \exp(My) \tag{4.2}$$

for some matrix M in the Lie algebra of U .

The map $\gamma(y)$ is not periodic, but has monodromy $\mathcal{M}(\gamma) = \gamma(0)\gamma(1)^{-1} = e^M$ in U and the physically distinct reductions are classified by the conjugacy class of the monodromy [23]. In the full theory in which all massive modes are kept, U is typically broken to a discrete subgroup $U(\mathbb{Z})$. In order for $\Psi(x, y + 2\pi) = \mathcal{M}\Psi(x, y)$ to be well-defined, the monodromy \mathcal{M} must therefore be in the symmetry group $U(\mathbb{Z})$ [15, 25].

If the monodromy is in the geometric $GL(d; \mathbb{Z})$ sub-group then the reduction may be viewed as a specific class of the identified group manifold reductions of the previous sections. In this case the identified group manifold really is a topologically twisted torus fibration. If the monodromy is more general then the reduction cannot be given as a purely geometric construction as the monodromy (or transition functions between patches) may now be S or T dualities. For example, if the monodromy is an element of the T-duality group, then the string theory is only defined in a certain patch and we must consider the identification of a particular string theory as only possible locally. Globally, this picture must be generalised to include the string theory and its T-dual. In particular, the transition function would invert the radii of the circles on a torus and generate non-trivial B -fields - clearly a non-geometric operation. The introduction of Ramond fields and their associated D-branes on a background with a monodromy in the factorised duality subgroup of $O(d, d)$ leads to further startling features [11, 26]. For example, the dimension of the D-brane would not be globally defined as the transition function increases or decreases the dimension of the brane as one moves between patches [26, 27]. Another example is that of string theory on a Calabi-Yau manifold [15]. Here the backgrounds are permitted to have transition functions which are mirror symmetries. It should be noted that truncation to the lowest modes on a Calabi-Yau is not consistent so the detailed analysis in that case is expected to be more complicated.

The situation is even more drastic if the monodromy is an element of S-duality. In this case, the perturbative string theory picture can only be used locally. In this section such reductions of IIB supergravity with S-duality twists are investigated at the supergravity level. Such reductions have also been investigated in [11, 28].

4.1 Reductions with S-duality twists

S-duality is non-perturbative in the string coupling g_s , mixing Ramond and Neveu-Schwarz fields, and therefore cannot be given a worldsheet interpretation but there is compelling evidence to believe that this symmetry is an exact symmetry of the full non-perturbative theory [29].

The reduction to nine dimensions of the bosonic sector of IIB supergravity on a circle with an S-duality twist will be considered. The ten-dimensional IIB Lagrangian, written in a manifestly $SL(2)$ invariant form was given by (3.2). Consider the reduction ansatz

$$ds_{D+1}^2 = e^{2\alpha\varphi} ds_D^2 + e^{2\beta\varphi} (dy + A)^2 \tag{4.3}$$

where y parameterises the circle direction. The duality twist ansatz (3.8), (4.1) and (4.2) to reduce the fields of the IIB theory on circle with S-duality twist are

$$\widehat{\mathcal{K}}^a_b(x, y) = e^{(M^t)^a_c y} \mathcal{K}^c_d(x) e^{M^d_b y} \quad \widehat{B}^a_{(2)}(x, y) = e^{M^a_b y} \left(B^b_{(2)}(x) + B^b_{(1)}(x) \wedge \nu \right) \tag{4.4}$$

where M is a twist matrix in the Lie algebra of $SL(2)$ and M^t its transpose. The reduced scalar Lagrangian is then⁸

$$\mathcal{L}_{\mathcal{K}} = \frac{1}{4} Tr (*DK \wedge DK^{-1}) - \frac{1}{2} e^{-2(\alpha+\beta)\varphi} Tr (M^2 + M^T \mathcal{K} M \mathcal{K}^{-1}) * 1 \tag{4.5}$$

⁸We have made use of the fact that $Tr(M^t \cdot M^t) = Tr(M \cdot M)$ in the potential.

where the covariant derivatives are

$$\begin{aligned} D_i \mathcal{K}^a_b &= \partial_i \mathcal{K}^a_b - (M^T \mathcal{K} + \mathcal{K} M)^a_b A_i \\ D_i (\mathcal{K}^{-1})^a_b &= \partial_i (\mathcal{K}^{-1})^a_b + (\mathcal{K}^{-1} M^T + M \mathcal{K}^{-1})^a_b A_i \end{aligned} \quad (4.6)$$

The reduction of the two form field strength term

$$\mathcal{L}_{\widehat{H}} = -\frac{1}{2} \mathcal{K}_{ab} * \widehat{H}_{(3)}^a \wedge \widehat{H}_{(3)}^b \quad (4.7)$$

gives the low energy effective term

$$\mathcal{L}_H = -\frac{1}{2} \mathcal{K}_{ab} e^{-4\alpha\varphi} * H_{(3)}^a \wedge H_{(3)}^b - \frac{1}{2} \mathcal{K}_{ab} e^{-2(\alpha+\beta)\varphi} * H_{(2)}^a \wedge H_{(2)}^b \quad (4.8)$$

where the reduced field strengths are

$$\begin{aligned} H_{(3)}^a &= dB_{(2)}^a + M^a_b B_{(2)}^b \wedge A - B_{(1)}^a \wedge F \\ &= DB_{(2)}^a - B_{(1)}^a \wedge F \\ H_{(2)}^a &= dB_{(1)}^a - M^a_b B_{(1)}^b \wedge A - M^a_b B_{(2)}^b \\ &= DB_{(1)}^a - M^a_b B_{(2)}^b \\ F &= dA \end{aligned} \quad (4.9)$$

with the Bianchi identities

$$DH_{(3)}^a = 0 \quad DH_{(2)}^a = 0 \quad dF = 0 \quad (4.10)$$

The self dual five form field strength, although a singlet under the S-duality transformation, still has a non-trivial deformation in the Scherk-Schwarz ansatz coming from the Chern-Simons terms. The five form term in the action is

$$\mathcal{L}_{\widehat{G}} = -\frac{1}{4} * \widehat{G}_{(5)} \wedge \widehat{G}_{(5)} \quad (4.11)$$

which reduces to

$$\mathcal{L}_G = -\frac{1}{4} e^{-8\alpha\varphi} * G_{(5)} \wedge G_{(5)} - \frac{1}{4} e^{-(2\alpha+\beta)\varphi} * G_{(4)} \wedge G_{(4)} \quad (4.12)$$

where

$$\begin{aligned} G_{(5)} &= dC_{(4)} - C_{(3)} \wedge F + \frac{1}{2} \epsilon_{ab} B_{(2)}^a \wedge H_{(3)}^b \\ G_{(4)} &= dC_{(3)} - \frac{1}{2} \epsilon_{ab} \left(B_{(1)}^a \wedge H_{(3)}^b - B_{(2)}^a \wedge H_{(2)}^b \right) \end{aligned} \quad (4.13)$$

Finally, the Chern-Simons terms reduce to

$$\mathcal{L}_{cs} = -\frac{1}{4} \left(C_{(3)} \wedge H_{(3)}^a \wedge H_{(3)}^b + 2C_{(4)} \wedge H_{(3)}^a \wedge H_{(2)}^b \right) \quad (4.14)$$

The self duality constraint $\widehat{G}_{(5)} = * \widehat{G}_{(5)}$ reduces to a relationship between $\widehat{G}_{(5)}$ and $\widehat{G}_{(4)}$ and must be imposed on the equations of motion of the reduced theory.

4.2 Gauge Symmetry and its Breaking

In this section the symmetries of the reduced Lagrangian are studied and the symmetry breaking and mass mechanisms involved are analysed.

4.2.1 Antisymmetric tensor transformations

In ten dimensions the three form field strength $H_{(3)}^a$ is invariant under the abelian anti-symmetric tensor transformation of the potential $\delta\widehat{B}_{(2)}^a = d\widehat{\lambda}_{(1)}^a$. The four form potential transforms to compensate for the transformation of the Chern-Simons term in the field strength $\widehat{G}_{(5)}$ as

$$\delta\widehat{C}_{(4)} = -\frac{1}{2}\epsilon_{ab}\widehat{\lambda}_{(1)}^a \wedge \widehat{H}_{(3)}^b \quad (4.15)$$

The combined transformations of the four-form and two-form potentials leave the five-form field strength, $\widehat{G}_{(5)}$, invariant. Under the S-duality twisted reduction considered in the preceding section, the reduced potentials transform as

$$\begin{aligned} \delta B_{(2)}^a &= d\lambda_{(1)}^a - M^a{}_b \lambda_{(1)}^b \wedge A + \lambda_{(0)}^a F \\ \delta B_{(1)}^a &= d\lambda_{(0)}^a + M^a{}_b \lambda_{(0)}^b \wedge A + M^a{}_b \lambda_{(1)}^b \\ \delta C_{(4)} &= -\frac{1}{2}\epsilon_{ab}\lambda_{(1)}^a \wedge H_{(3)}^b \\ \delta C_{(3)} &= -\frac{1}{2}\epsilon_{ab} \left(\lambda_{(1)}^a \wedge H_{(2)}^b - \lambda_{(0)}^a H_{(3)}^b \right) \end{aligned} \quad (4.16)$$

The transformation $\delta B_{(1)}^a = M^a{}_b \lambda_{(1)}^b + \dots$ is a shift symmetry, i.e. it is non-linear realisation of the symmetry group and will not be preserved by any vacuum of the theory. A massive two-form $\check{B}_{(2)}^a$ may be defined

$$\check{B}_{(2)}^a = B_{(2)}^a - (M^{-1})^a{}_b D B_{(1)}^b \quad (4.17)$$

where $B_{(2)}^a$ has eaten $B_{(1)}^a$ to become massive and $\check{B}_{(2)}^a$ is a singlet of the symmetry transformations (4.16). The redefinition (4.17) is dependent on the existence of the inverse $(M^{-1})^a{}_b$. It will be shown that that this is not always the case and that care must be taken in defining the massive two form fields (4.17). For now it will be assumed that the mass matrix is chosen such that $(M^{-1})^a{}_b$ exists. Applying the field redefinitions

$$\begin{aligned} \check{C}_{(3)} &= C_{(3)} + \frac{1}{2}\epsilon_{ab}(M^{-1})^a{}_c B_{(1)}^c \wedge H_{(2)}^b \\ \check{C}_{(4)} &= C_{(4)} + \frac{1}{2}\epsilon_{ab}(M^{-1})^a{}_c B_{(1)}^c \wedge H_{(3)}^b \end{aligned} \quad (4.18)$$

the field strengths become

$$\begin{aligned} G_{(5)} &= d\check{C}_{(4)} - \check{C}_{(3)} \wedge F + \frac{1}{2}\epsilon_{ab}\check{B}_{(2)}^a \wedge D\check{B}_{(2)}^b \\ G_{(4)} &= d\check{C}_{(3)} - \frac{1}{2}M_{ab}\check{B}_{(2)}^a \wedge \check{B}_{(2)}^b \\ H_{(3)}^a &= D\check{B}_{(2)}^a \\ H_{(2)}^a &= -M^a{}_b \check{B}_{(2)}^b \end{aligned} \quad (4.19)$$

where $M_{ab} = \epsilon_{(a|c} M^c_{|b)}$. $B_{(1)}^a$ is eaten by $\check{B}_{(2)}^a$ and completely drops out of all of the field equations. The $\check{C}_{(3)}$ and $\check{C}_{(4)}$ fields remain massless and charged under the abelian gauge symmetry generated by the transformations

$$\delta\check{C}_{(3)} = d\Lambda_{(2)} \quad \delta\check{C}_{(4)} = d\Lambda_{(3)} \quad (4.20)$$

The Lagrangian of the reduced theory is then

$$\begin{aligned} \mathcal{L}_9 = & R * 1 - \frac{1}{2} * d\varphi \wedge d\varphi - \frac{1}{2} e^{2(\beta-\alpha)\varphi} * F \wedge F + \frac{1}{4} Tr (*DK \wedge DK^{-1}) \\ & - \frac{1}{4} e^{-8\alpha\varphi} * G_{(5)} \wedge G_{(5)} - \frac{1}{4} e^{-(2\alpha+\beta)\varphi} * G_{(4)} \wedge G_{(4)} \\ & - \frac{1}{2} \mathcal{K}_{ab} e^{-4\alpha\varphi} * D\check{B}_{(2)}^a \wedge D\check{B}_{(2)}^b - \frac{1}{2} \mathcal{K}_{ab} e^{-2(\alpha+\beta)\varphi} M^a_c M^b_d * \check{B}_{(2)}^c \wedge \check{B}_{(2)}^d \\ & - \frac{1}{4} \epsilon_{ab} \left(\check{C}_{(3)} \wedge H_{(3)}^a \wedge H_{(3)}^b + 2\check{C}_{(4)} \wedge H_{(3)}^a \wedge H_{(2)}^b \right) \\ & - \frac{1}{2} e^{-2(\alpha+\beta)\varphi} Tr (M^2 + M^t \mathcal{K} M \mathcal{K}^{-1}) * 1 \end{aligned} \quad (4.21)$$

In some cases M^a_b may not be invertible. An example is the reduction with parabolic twist where

$$M_p = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \quad m \in \mathbb{Z} \quad (4.22)$$

This mass matrix has no inverse so one must be careful in defining the massive two form (4.17). For such non-invertible matrices one may always choose a basis such that the mass matrix takes the form

$$M = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \quad (4.23)$$

In this basis the potentials are written as

$$B_{(1)}^a = \begin{pmatrix} B'_{(1)} \\ \bar{B}_{(1)} \end{pmatrix} \quad B_{(2)}^a = \begin{pmatrix} B'_{(2)} \\ \bar{B}_{(2)} \end{pmatrix} \quad (4.24)$$

It is then possible to identify a massive two-form

$$\check{B}_{(2)} = B'_{(2)} - M^{-1} D B'_{(1)} \quad (4.25)$$

whilst $\bar{B}_{(1)}$ and $\bar{B}_{(2)}$ remain massless.

4.2.2 Internal diffeomorphism and fixed points of the twist

In addition to the antisymmetric tensor transformations, the reduced theory has a U(1) gauge symmetry originating from diffeomorphisms $y \rightarrow y - \omega(x)$ along the compactification circle. The reduced fields transform as

$$\begin{aligned} \delta A &= d\omega \\ \delta B_{(1)}^a &= M^a_b B_{(1)}^b \omega \\ \delta B_{(2)}^a &= M^a_b B_{(2)}^b \omega \\ \delta \mathcal{K}_{ab} &= -2\mathcal{K}_{(a|c} M^c_{|b)} \omega \end{aligned} \quad (4.26)$$

where $\omega = \omega(x)$. For a given expectation value of the scalars $\langle \mathcal{K} \rangle_{ab} = \overline{\mathcal{K}}_{ab}$ this symmetry will be broken unless

$$\langle \delta \mathcal{K} \rangle_{ab} = -2\overline{\mathcal{K}}_{(a|c} M^c{}_{|b)} \omega = 0 \tag{4.27}$$

If this is the case, then the graviphotons are massless, as may be seen from the $D\mathcal{K}$ term in the Lagrangian (4.21)

$$\mathcal{L}_9 = \frac{1}{4} Tr \left(*D\overline{\mathcal{K}} \wedge D\overline{\mathcal{K}}^{-1} \right) + \dots = -\frac{1}{2} \mu^2 * A \wedge A + \dots \tag{4.28}$$

where the mass μ is given by

$$\mu^2 = Tr \left(M^2 + M^t \overline{\mathcal{K}} M \overline{\mathcal{K}}^{-1} \right) \tag{4.29}$$

This μ^2 term is proportional to the scalar potential of the reduction and therefore the graviphotons will be massless at the minima of the potential. The minima of the potential for such reductions was studied in [15] where it was shown that the potential is zero only for elliptic twists, an argument that is reviewed here. The moduli matrix may be written in terms of the $SL(2)/SO(2)$ zweibein \mathcal{V} as $\mathcal{K}_{ab} = \mathcal{V}_a^\alpha \delta_{\alpha\beta} \mathcal{V}^\beta_b$ where

$$\mathcal{V} = e^{\frac{1}{2}\phi} \begin{pmatrix} 1 & C_{(0)} \\ 0 & e^{-\phi} \end{pmatrix} \tag{4.30}$$

and $\alpha, \beta = 1, 2$ are $SO(2)$ matrix representation indices. The vanishing point for the potential and graviphoton mass occurs when the complex structure has the vacuum value $\langle \mathcal{V} \rangle = \mathcal{V}_0$ such that the twist matrix M is equivalent to a matrix R in the Lie algebra of $SO(2)$ up to a conjugation by \mathcal{V}_0

$$M = \mathcal{V}_0^{-1} R \mathcal{V}_0 \tag{4.31}$$

This may be seen as follows. Setting $\overline{\mathcal{K}} = \mathcal{V}_0^t \mathcal{V}_0$ and $M = \mathcal{V}_0^{-1} R \mathcal{V}_0$ the potential may be written as [15]

$$V \propto \mu^2 = \frac{1}{2} Tr (Y^2) \tag{4.32}$$

where $Y = R + R^t$. Since R is in the Lie algebra of $SO(2)$, $Y = 0$ and the potential vanishes and the graviphotons are massless. As recognised in [15] this is to be expected as such a choice of vacuum is a fixed point of the twist action and therefore will have no effect on the field theory. For this choice of scalars (4.27) can be written

$$\begin{aligned} \langle \delta \mathcal{K} \rangle_{ab} &= -(\mathcal{V}_0)_a^\alpha R_{\alpha\beta} (\mathcal{V}_0)^\beta_b - (\mathcal{V}_0)_b^\alpha R_{\alpha\beta} (\mathcal{V}_0)^\beta_a \\ &= -2(\mathcal{V}_0)_a^\alpha R_{(\alpha\beta)} (\mathcal{V}_0)^\beta_b \end{aligned} \tag{4.33}$$

The right hand side of (4.33) vanishes as the generator $R_{\alpha\beta}$ of $SO(2)$ is antisymmetric. Therefore a choice of vacuum will generally break the $U(1)$ isometry group unless the twist is in the elliptic conjugacy class.

4.3 Non-geometric twists and F-theory

F-Theory is formally a twelve dimensional theory which when reduced on T^2 gives a theory whose truncation to the massless sector gives the IIB supergravity. The S-duality of the IIB theory is then described geometrically as the mapping class group of the T^2 fibre for which the axio-dilaton $\tau = C_{(0)} + ie^{-\phi}$ is the complex structure. This, apparently redundant, description of the IIB theory becomes useful when one considers compactifications of F-Theory on spaces that have a T^2 fibration that is not trivial [30]. The relevance of this picture here is that in some cases it may be used to give a geometrical interpretation to otherwise non-geometric duality twist backgrounds.

Consider for example the $SL(2, \mathbb{Z})$ U-duality of the IIB string theory [25]. Reducing from 10 to 9 dimensions on a circle with monodromy in $SL(2, \mathbb{Z})$ investigated in the previous section and also [23, 24, 31, 32]. As the $SL(2, \mathbb{Z})$ symmetry is not geometric, this cannot be realised as a compactification in the conventional sense. However, it can be realised as a ‘compactification’ of F-theory on the twisted torus corresponding to a T^2 bundle over S^1 with $SL(2, \mathbb{Z})$ monodromy [23]. For example, the case of an elliptic twist with vanishing potential discussed in the last section may be thought of as a reduction of F-Theory on an orbifold [15].

This further extends the notion of a non-geometric background to a non-perturbative background where one must cover the internal circle with at least two patches and where the transition functions between patches are S-dualities. If these Scherk-Schwarz reductions lift to solutions of the full M-Theory one must accept that, even at weak coupling, perturbative string theory can at best describe only the local physics of such solutions.

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A. Bianchi identities and field strengths

The reduced field strengths are

$$\begin{aligned}
 \mathcal{H}_{(3)}^a &= dB_{(2)}^a + B_{(1)m}^a \wedge F^m + \frac{1}{6} M_{mnp}^a A^m \wedge A^n \wedge A^p \\
 \mathcal{H}_{(2)m}^a &= DB_{(1)m}^a + B_{(0)mn}^a F^n + \frac{1}{2} M_{mnp}^a A^n \wedge A^p \\
 \mathcal{H}_{(1)mn}^a &= DB_{(0)mn}^a + f_{mn}^p B_{(1)p}^a + M_{mnp}^a A^p \\
 \mathcal{H}_{(0)mnp}^a &= 3B_{(0)[m]q}^a f_{[np]}^q + M_{mnp}^a
 \end{aligned} \tag{A.1}$$

and for the five form field strength

$$\begin{aligned}
 G_{(5)} &= dC_{(4)} + C_{(3)m} \wedge F^m + \frac{1}{120} K_{mnpqt} A^m \wedge A^n \wedge A^p \wedge A^q \wedge A^t \\
 &\quad + \frac{1}{2} \epsilon_{ab} B_{(2)}^a \wedge H_{(3)}^b - \frac{1}{6} \epsilon_{ab} M_{mnp}^a B_{(2)}^b \wedge A^m \wedge A^n \wedge A^p
 \end{aligned}$$

$$\begin{aligned}
 G_{(4)m} &= DC_{(3)m} + C_{(2)mn} \wedge F^n + \frac{1}{24} K_{mnpqt} A^n \wedge A^p \wedge A^q \wedge A^t \\
 &\quad + \frac{1}{2} \epsilon_{ab} B_{(2)}^a \wedge H_{(2)m}^b - \frac{1}{2} \epsilon_{ab} B_{(2)}^a \wedge H_{(3)}^b \\
 &\quad + \frac{1}{6} \epsilon_{ab} M_{qnp}^a B_{(1)m}^b \wedge A^n \wedge A^p \wedge A^q - \frac{1}{2} \epsilon_{ab} M_{mnp}^a B_{(2)}^b \wedge A^n \wedge A^p \\
 G_{(3)mn} &= DC_{(2)mn} + C_{(2)p} f_{mn}^p + C_{(1)mnp} \wedge F^p + \frac{1}{6} K_{mnpqt} A^p \wedge A^q \wedge A^t \\
 &\quad + \frac{1}{2} \epsilon_{ab} B_{(0)mn}^a H_{(3)}^b + \frac{1}{2} \epsilon_{ab} B_{(1)m}^a \wedge H_{(2)n}^b + \frac{1}{2} \epsilon_{ab} B_{(2)}^a \wedge H_{(1)mn}^b \\
 &\quad - \epsilon_{ab} \left(M_{qtp}^a B_{(0)mn}^b A^q \wedge A^t + M_{mpq}^a B_{(1)n}^b \wedge A^q + M_{mnp}^a B_{(2)}^b \right) \wedge A^p \\
 G_{(2)mnp} &= DC_{(1)mnp} + C_{(0)mnpq} F^q + \frac{1}{2} K_{mnpqt} A^q \wedge A^t \\
 &\quad + \frac{3}{2} \epsilon_{ab} B_{(2)}^a B_{(0)[m|q}^b f_{np]}^q + \frac{3}{2} \epsilon_{ab} B_{(0)mn}^a H_{(2)p}^b - \frac{3}{2} \epsilon_{ab} B_{(1)m}^a \wedge H_{(1)np}^b \\
 &\quad - \epsilon_{ab} \left(M_{mnp}^a B_{(2)}^b - 3M_{mnq}^a B_{(1)p}^b \wedge A^q + \frac{3}{2} M_{pqt}^a B_{(0)mn}^b A^q \wedge A^t \right) \\
 G_{(1)mnpq} &= DC_{(0)mnpq} + 6C_{(1)pqt} f_{mn}^t + K_{mnpqt} A^t \\
 &\quad + 3\epsilon_{ab} B_{(0)mn}^a H_{(1)pq}^b + 6\epsilon_{ab} B_{(1)m}^a B_{(0)[n|t}^b f_{pq]}^t \\
 &\quad - \epsilon_{ab} \left(6M_{mnt}^a B_{(0)pq}^b \wedge A^t - 4M_{mnp}^a B_{(1)q}^b \right) \\
 G_{(0)mnpqt} &= -2C_{(0)spqt} f_{mn}^s + K_{mnpqt} \\
 &\quad + 30\epsilon_{ab} B_{(0)mn}^a B_{(0)[p|s}^b f_{qt]}^s - 10\epsilon_{ab} M_{mnp}^a B_{(0)qt}^b
 \end{aligned} \tag{A.2}$$

where the G_R covariant derivatives are

$$D\psi_{(p)m_1 m_2 \dots m_q} = d\psi_{(p)m_1 m_2 \dots m_q} + (-)^p \psi_{(p)[m_1 m_2 \dots m_{q-1} | n} f_{|m_q]p}^n \wedge A^p \tag{A.3}$$

The reduced Bianchi identities for the self-dual five form are

$$\begin{aligned}
 dG_{(5)} - G_{(4)m} \wedge F^m &= \frac{1}{2} \epsilon_{ab} \mathcal{H}_{(3)}^a \wedge \mathcal{H}_{(3)}^b \\
 DG_{(4)m} - G_{(3)mn} \wedge F^n &= \epsilon_{ab} \mathcal{H}_{(3)}^a \wedge \mathcal{H}_{(2)m}^b \\
 DG_{(3)mn} - f_{mn}^p G_{(4)p} - G_{(2)mnp} \wedge F^p &= \epsilon_{ab} \left(\mathcal{H}_{(3)}^a \wedge \mathcal{H}_{(1)mn}^b + \mathcal{H}_{(2)m}^a \wedge H_{(2)np}^b \right) \\
 DG_{(2)mnp} + O_{mnp}^{qt} G_{(3)qt} - G_{(1)mnpq} \wedge F^q &= \epsilon_{ab} \left(\mathcal{H}_{(3)}^a \wedge \mathcal{H}_{(0)mnp}^b - 3\mathcal{H}_{(2)m}^a \wedge H_{(2)np}^b \right) \\
 DG_{(1)mnpq} - O_{mnpq}^{tsl} G_{(2)tsl} - G_{(0)mnpqt} \wedge F^t &= \epsilon_{ab} \left(4\mathcal{H}_{(2)[m}^a \mathcal{H}_{(0)npq]}^b + 3\mathcal{H}_{(1)[mn}^a \wedge \mathcal{H}_{(0)pqt]}^b \right) \\
 DG_{(0)mnpqt} + O_{mnpqt}^{slj} G_{(1)slj} &= 10\epsilon_{ab} \mathcal{H}_{(1)[mn}^a \mathcal{H}_{(0)pqt]}^b \\
 O_{mnpqts}^{lijk} G_{(0)lijkh} &= 10\epsilon_{ab} \mathcal{H}_{(0)[mnp}^a \mathcal{H}_{(0)qts]}^b
 \end{aligned} \tag{A.4}$$

and for the three form

$$\begin{aligned}
 d\mathcal{H}_{(3)}^a + \mathcal{H}_{(2)m}^a \wedge F^m &= 0 \\
 D\mathcal{H}_{(2)m}^a + \mathcal{H}_{(1)mn}^a \wedge F^n &= 0 \\
 D\mathcal{H}_{(1)mn}^a + \mathcal{H}_{(0)mnp}^a F^p &= 0 \\
 D\mathcal{H}_{(0)mnp}^a &= 0
 \end{aligned} \tag{A.5}$$

B. Right action of the group manifold

The identified group manifold $\mathcal{X} = G/\Gamma$ inherits the right action of the group G_R on G . The calculation of how the reduced fields transform under G_R is somewhat involved and it is helpful to clarify the discussion by first considering the simpler case of the two form transformation. The action of G_R on the two-form is generated by the Lie derivative \mathcal{L}_ω

$$\mathcal{L}_\omega(\widehat{\mathcal{B}}_{(2)}^a) = \mathcal{L}_\omega(\widehat{B}_{(2)}^a) + \mathcal{L}_\omega(\varpi_{(2)}^a) = 0 \quad (\text{B.1})$$

where $\omega = \omega(x)$ is the parameter associated to the right action on the group manifold G . Using the fact that the Lie derivative may be written $\mathcal{L}_\omega = \iota_\omega d + d\iota_\omega$, $\mathcal{L}_\omega(\nu^m) = -\nu^n f_{np}{}^m \omega^p$ and choosing the convention $\iota_\omega \sigma^m = -\omega^m$, the transformations in (B.1) imply

$$\mathcal{L}_\omega(\widehat{B}_{(2)}^a) = -\mathcal{L}_\omega(\varpi_{(2)}^a) = \frac{1}{2} M_{mnp}{}^a \omega^p \sigma^m \wedge \sigma^n + d\Xi_{(1)}^a \quad (\text{B.2})$$

where $\Xi_{(1)}^a \equiv \iota_\omega \varpi_{(2)}^a$. $\widehat{B}_{(2)}^a$ is in addition transformed by $\delta(\Xi)_Y \widehat{B}_{(2)}^a = -d\Xi_{(1)}^a$ defining a gauge transformation $\delta_Z(\omega)$ which is independent of the internal coordinates y^i

$$\delta_Z(\omega) \widehat{B}_{(2)}^a = \frac{1}{2} M_{mnp}{}^a \omega^p \sigma^m \wedge \sigma^n \quad (\text{B.3})$$

i.e.

$$\delta_Z(\omega) \widehat{\mathcal{B}}_{(2)}^a = \mathcal{L}_\omega(\widehat{\mathcal{B}}_{(2)}^a) - \delta_Y(\Xi) \widehat{\mathcal{B}}_{(2)}^a = -\delta_Y(\Xi) \widehat{\mathcal{B}}_{(2)}^a \quad (\text{B.4})$$

where the second equality comes from the fact that the $\widehat{\mathcal{B}}_{(2)}^a$ is invariant under general coordinate transformations generated by \mathcal{L}_ω . The reduced components of $\widehat{\mathcal{B}}_{(2)}^a$ transform as

$$\begin{aligned} \delta_Z(\omega) B_{(2)}^a &= \frac{1}{2} M_{mnp}{}^a \omega^p A^m \wedge A^n \\ \delta_Z(\omega) B_{(1)m}^a &= B_{(1)n}^a f_{mp}{}^n \omega^p - M_{mnp}{}^a \omega^p A^n \\ \delta_Z(\omega) B_{(0)mn}^a &= 2B_{(1)[m|p}^a f_{|n]q}{}^p \omega^q + M_{mnp}{}^a \omega^p \end{aligned} \quad (\text{B.5})$$

The corresponding transformation for the four form potential requires more care as the flux for this potential (3.12) is more complicated. The symmetry transformation of interest is that generated by $\delta_Z(\omega)$ rather than \mathcal{L}_ω and are defined by $\mathcal{L}_\omega(G_{(5)}) = 0$, which requires the four form potential to transform as

$$\begin{aligned} \delta_Z(\omega) \widehat{\mathcal{C}}_{(4)} &= \mathcal{L}_\omega(\widehat{\mathcal{C}}_{(4)}) - \delta_Y(\Xi) \widehat{\mathcal{C}}_{(4)} \\ &= -\delta_Y(\Xi) \widehat{\mathcal{C}}_{(4)} \\ &= \frac{1}{2} \epsilon_{ab} \Xi_{(1)}^a \wedge \widehat{\mathcal{H}}_{(3)}^b + d\chi_{(3)} \end{aligned} \quad (\text{B.6})$$

where the $\delta_Y(\Xi)$ transformation is only defined up to the arbitrary total derivative $d\chi_{(3)}$ and it must be stressed that $\widehat{\mathcal{C}}_{(4)}$ is invariant under the diffeomorphism transformation \mathcal{L}_ω , but not $\delta_Z(\omega)$. Combining (3.12) and (B.6), the symmetry transformation of interest is

$$\delta_Z(\omega) \widehat{\mathcal{S}}_{(4)} = \delta_Z(\omega) \left(\frac{1}{2} \varpi_{(2)}^a \wedge \widehat{B}_{(2)}^b - \varpi_{(4)} \right) + \frac{1}{2} \epsilon_{ab} \Xi_{(1)}^a \wedge \widehat{\mathcal{H}}_{(3)}^b + d\chi_{(3)} \quad (\text{B.7})$$

It will be shown that, although $\mathcal{L}_\omega \widehat{S}$ is not y -independent, $\delta_Z(\omega) \widehat{S}$ is. The transformation $\delta_Z(\omega) d\varpi_{(4)} = \mathcal{L}_\omega(d\varpi_{(4)})$ will be calculated first.⁹ Consider,

$$\begin{aligned} \mathcal{L}_\omega(\mathcal{K}_{(5)}) &= (i_\omega d + di_\omega) \frac{1}{120} K_{mnpqt} \sigma^m \wedge \sigma^n \wedge \sigma^p \wedge \sigma^q \wedge \sigma^t \\ &= -\frac{1}{24} K_{[mnpq]t} f_{[l]s}^t \omega^s \sigma^m \wedge \sigma^n \wedge \sigma^p \wedge \sigma^q \wedge \sigma^l \\ &\quad - \frac{1}{24} K_{mnpqt} d\omega^t \wedge \sigma^m \wedge \sigma^n \wedge \sigma^p \wedge \sigma^q \end{aligned} \quad (\text{B.8})$$

and the transformation

$$\mathcal{L}_\omega(\varpi_{(2)}^a) = -\frac{1}{2} M_{mnp}{}^a \omega^p \sigma^m \wedge \sigma^n + d\Xi_{(1)}^a \quad (\text{B.9})$$

Putting together (3.13), (B.8) and (B.9) the transformation of $d\varpi_{(4)}$ is therefore

$$\begin{aligned} \mathcal{L}_\omega(d\varpi_{(4)}) &= \mathcal{L}_\omega \left(-\frac{1}{2} \epsilon_{ab} \varpi_{(2)}^a \wedge d\varpi_{(2)}^b + \mathcal{K}_{(5)} \right) \\ &= -\frac{1}{12} \epsilon_{ab} \left(-\frac{1}{2} M_{[mn]p}{}^a \omega^p \sigma^m \wedge \sigma^n + d\Xi_{(1)}^a \right) \wedge M_{[qts]}{}^b \sigma^q \wedge \sigma^t \wedge \sigma^s \\ &\quad + \frac{1}{4} \epsilon_{ab} \varpi_{(2)}^a \wedge M_{mnp}{}^b d(\omega^p \sigma^m \wedge \sigma^n) + \mathcal{L}_\omega(\mathcal{K}_{(5)}) \end{aligned} \quad (\text{B.10})$$

which can be written as

$$\mathcal{L}_\omega(d\varpi_{(4)}) = -d\theta_{(4)} - \frac{1}{2} \epsilon_{ab} d\varpi_{(2)}^a \wedge M_{mnp}{}^b \omega^p \sigma^m \wedge \sigma^n + \mathcal{L}_\omega(\mathcal{K}_{(5)}) \quad (\text{B.11})$$

where

$$\theta_{(4)} = \frac{1}{2} \epsilon_{ab} \varpi_{(2)}^a \wedge \left(-\frac{1}{2} M_{mnp}{}^b \omega^p \sigma^m \wedge \sigma^n - d\Xi_{(1)}^b \right) \quad (\text{B.12})$$

Now consider the second term in the above expression (B.11)

$$\frac{1}{2} \epsilon_{ab} d\varpi_{(2)}^a \wedge M_{mnp}{}^b \omega^p \sigma^m \wedge \sigma^n = \frac{1}{12} \epsilon_{ab} M_{[mnp]}{}^a M_{[qts]}{}^b \omega^s \sigma^m \wedge \sigma^n \wedge \sigma^p \wedge \sigma^q \wedge \sigma^t \quad (\text{B.13})$$

Using the fact that $\epsilon_{ab} M_{[mnp]}{}^a M_{[qts]}{}^b = \epsilon_{ab} M_{[mnp]}{}^a M_{[qts]}{}^b$, (B.13) may be written as

$$\frac{1}{12} \epsilon_{ab} M_{[mnp]}{}^a M_{[qts]}{}^b \omega^s \sigma^m \wedge \sigma^n \wedge \sigma^p \wedge \sigma^q \wedge \sigma^t \quad (\text{B.14})$$

and now making use of the identity $2\epsilon_{ab} M_{[mnp]}{}^a M_{[qts]}{}^b + 3K_{[mnpq]l} f_{[ts]}^l = 0$ (3.17) to write (B.14) as

$$\begin{aligned} \frac{1}{2} \epsilon_{ab} d\varpi_{(2)}^a \wedge M_{mnp}{}^b \omega^p \sigma^m \wedge \sigma^n &= -\frac{1}{24} K_{[mnpq]l} f_{[ts]}^l \omega^s \sigma^m \wedge \sigma^n \wedge \sigma^p \wedge \sigma^q \wedge \sigma^t \\ &\quad + \frac{1}{12} K_{[mnp]sl} f_{[tq]}^l \omega^s \sigma^m \wedge \sigma^n \wedge \sigma^p \wedge \sigma^q \wedge \sigma^t \end{aligned} \quad (\text{B.15})$$

⁹In all the variations of the fluxes $\delta_Z(\omega)\varpi = \mathcal{L}_\omega(\varpi)$ as the fluxes are invariant under the δ_Y and δ_X transformations.

and substituting it into (B.10), the variation of $d\varpi_{(4)}$ becomes

$$\begin{aligned}
 \mathcal{L}_\omega(d\varpi_{(4)}) &= -d\theta_{(4)} - \frac{1}{2}\epsilon_{ab}d\varpi_{(2)}^a \wedge M_{mnp}{}^b \omega^p \sigma^m \wedge \sigma^n + \mathcal{L}_\omega(\mathcal{K}_{(5)}) \\
 &= -d\theta_{(4)} - \frac{1}{24}K_{mnpqt}d\omega^t \wedge \sigma^m \wedge \sigma^n \wedge \sigma^p \wedge \sigma^q \\
 &\quad + \frac{1}{12}K_{[mnp]qt}f_{[sl]}{}^t \omega^q \wedge \sigma^m \wedge \sigma^n \wedge \sigma^p \wedge \sigma^s \wedge \sigma^l \\
 &= d\left(-\theta_{(4)} - \frac{1}{24}K_{mnpqt}\omega^t \sigma^m \wedge \sigma^n \wedge \sigma^p \wedge \sigma^q\right)
 \end{aligned} \tag{B.16}$$

It is simple to show that, acting on the space of forms, $[\mathcal{L}_\omega, d] = 0$ and therefore the variation of $\varpi_{(4)}$ commutes with the total derivative so that the expression (B.16) can be integrated to give

$$\begin{aligned}
 \mathcal{L}_\omega(\varpi_{(4)}) &= -\theta_{(4)} - \frac{1}{24}K_{mnpqt}\omega^t \sigma^m \wedge \sigma^n \wedge \sigma^p \wedge \sigma^q + d\Omega_{(3)} \\
 &= -\frac{1}{2}\epsilon_{ab}\varpi_{(2)}^a \wedge \left(-\frac{1}{2}M_{mnp}{}^b \omega^p \sigma^m \wedge \sigma^n - d\Xi_{(1)}^b\right) \\
 &\quad - \frac{1}{24}K_{mnpqt}\omega^t \sigma^m \wedge \sigma^n \wedge \sigma^p \wedge \sigma^q + d\Omega_{(3)}
 \end{aligned} \tag{B.17}$$

for some arbitrary three form $\Omega_{(3)}$. Now that $\mathcal{L}_\omega(\varpi_{(4)})$ has been determined, the variation of the second term in (B.7) is considered

$$\begin{aligned}
 \delta_Z(\omega) \left(-\frac{1}{2}\epsilon_{ab}\varpi_{(2)}^a \wedge \widehat{B}_{(2)}^b\right) &= -\frac{1}{2}\epsilon_{ab}\mathcal{L}_\omega(\varpi_{(2)}^a) \wedge \widehat{B}_{(2)}^b - \frac{1}{2}\epsilon_{ab}\varpi_{(2)}^a \wedge \left(\delta_Z(\omega)\widehat{B}_{(2)}^b\right) \\
 &= -\frac{1}{2}\epsilon_{ab} \left(-\frac{1}{2}M_{mnp}{}^a \omega^p \sigma^m \wedge \sigma^n + d\Xi_{(1)}^a\right) \wedge \widehat{B}_{(2)}^b \\
 &\quad - \frac{1}{4}\epsilon_{ab}\varpi_{(2)}^a \wedge M_{mnp}{}^b \omega^p \sigma^m \wedge \sigma^n
 \end{aligned} \tag{B.18}$$

where a gauge transformation is incorporated into $\delta(\omega)\widehat{B}_{(2)}^b$ to give $\delta_Z(\omega)\widehat{B}_{(2)}^b$. Putting these results together gives

$$\begin{aligned}
 \delta_Z(\omega) \left(-\frac{1}{2}\epsilon_{ab}\varpi_{(2)}^a \wedge \widehat{B}_{(2)}^b + \varpi_{(4)}\right) &= \frac{1}{4}\epsilon_{ab}M_{mnp}{}^a \omega^p \sigma^m \wedge \sigma^n \wedge \widehat{B}_{(2)}^b \\
 &\quad - \frac{1}{2}\epsilon_{ab}d\Xi_{(1)}^a \wedge \widehat{B}_{(2)}^b + \frac{1}{2}\varpi_{(2)}^a \wedge d\Xi_{(2)}^b + d\Omega_{(3)} \\
 &\quad - \frac{1}{24}K_{mnpqt}\omega^t \sigma^m \wedge \sigma^n \wedge \sigma^p \wedge \sigma^q
 \end{aligned} \tag{B.19}$$

The first two terms in the last line can be written as

$$\frac{1}{2}\epsilon_{ab}d\Xi_{(1)}^a \wedge \widehat{B}_{(2)}^b - \frac{1}{2}\varpi_{(2)}^a \wedge d\Xi_{(2)}^b = d\left(\frac{1}{2}\epsilon_{ab}\Xi_{(1)}^a \wedge \widehat{B}_{(2)}^b\right) + \frac{1}{2}\epsilon_{ab}\Xi_{(1)}^a \wedge \mathcal{H}_{(3)}^b \tag{B.20}$$

and the first term in (B.20) can be removed by choosing an appropriate value for $\Omega_{(3)}$ in (B.19) and the last term of (B.20) cancels with the $\delta_Y(\Xi)$ transformation of (B.7) to leave the transformation

$$\delta_Z(\omega)\widehat{S}_{(4)} = \frac{1}{4}\epsilon_{ab}M_{mnp}{}^a \omega^p \sigma^m \wedge \sigma^n \wedge \widehat{B}_{(2)}^b - \frac{1}{24}K_{mnpqt}\omega^t \sigma^m \wedge \sigma^n \wedge \sigma^p \wedge \sigma^q \tag{B.21}$$

Substituting the Scherk-Schwarz ansatz for the potentials (3.20) and (3.21) into (B.21), the gauge transformations of the potentials are found to be

$$\begin{aligned}
\delta_Z(\omega^m)S_{(4)} &= \frac{1}{4}\epsilon_{ab}M_{mnp}{}^a\omega^p B_{(2)}^b \wedge A^m \wedge A^n - \frac{1}{24}K_{mnpqt}\omega^t A^m \wedge A^n \wedge A^p \wedge A^q \\
\delta_Z(\omega^m)S_{(3)m} &= S_{(3)n}f_{mp}{}^n\omega^p + \frac{1}{4}\epsilon_{ab}M_{npq}{}^a\omega^q A^n \wedge A^p \wedge B_{(1)m}^b \\
&\quad + \frac{1}{2}\epsilon_{ab}M_{mnp}{}^a\omega^p A^n \wedge B_{(2)}^b + \frac{1}{6}K_{mnpqt}\omega^t A^n \wedge A^p \wedge A^q \\
\delta_Z(\omega^m)S_{(2)mn} &= 2S_{(2)[m|p}f_{|n]q}{}^p\omega^q + \frac{1}{4}\epsilon_{ab}M_{pqt}{}^a\omega^t A^p \wedge A^q B_{(0)mn}^b + \epsilon_{ab}M_{mpq}{}^a\omega^p A^q \wedge B_{(1)n}^b \\
&\quad + \frac{1}{2}\epsilon_{ab}M_{mnp}{}^a\omega^p B_{(2)}^b - \frac{1}{2}K_{mnpqt}\omega^t A^p \wedge A^q \\
\delta_Z(\omega^m)S_{(1)mnp} &= 3S_{(1)[mn|q}f_{|p]t}{}^q\omega^t + \frac{3}{2}\epsilon_{ab}M_{mnq}{}^a\omega^q B_{(1)p}^b - \frac{3}{2}\epsilon_{ab}M_{mqt}{}^a\omega^t B_{(0)np}^b A^q \\
&\quad + K_{mnpqt}\omega^t A^q \\
\delta_Z(\omega^m)S_{(0)mnpq} &= 4S_{(0)[mnp|t}f_{|q]s}{}^t\omega^s + 3\epsilon_{ab}M_{[mn|t}{}^a B_{(0)|pq]}^b\omega^t - K_{mnpqt}\omega^t
\end{aligned} \tag{B.22}$$

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